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Linear series on semistable curves.

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Abstract. For a semistable curve X of genus g, the number $h^0(X, L)$ is studied for line bundles L of degree d parametrized by the compactified Picard scheme. The theorem of Riemann is shown to hold. The theorem of Clifford is shown to hold in the following cases: X has two components; X is any semistable curve and $d = 0$ or $d = 2g - 2$; X is stable, free from separating nodes, and $d \leq 4$. These results are shown to be sharp. Applications to the Clifford index, to the combinatorial description of hyperelliptic curves, and to plane quintics are given.

Contents

1. Introduction and preliminaries

The dimension of complete linear series on singular curves is, in general, quite difficult to control. This is one of the reasons why several interesting degeneration problems about line bundles and linear series remain unsolved. For singular curves the Riemann-Roch theorem does not yield as strong information as for smooth curves, and several other classical theorems fail, as we shall illustrate.

On the other hand, it is well known that the Picard scheme of a singular curve tends to be too large, so that any good compactification of the generalized jacobian parametrizes only a distinguished subset of line bundles. At present time the geometric and functorial properties of the compactified Picard scheme are rather well understood, making it a natural place to study limits of line bundles and related problems.

This is the main theme of this paper, which investigates the dimension of complete linear series parametrized by the compactified Picard scheme of stable curves.

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They correspond to so-called balanced line bundles on semistable curves (defined in 2.1.1).

There exist other approaches to this type of questions. Some of them are by now considered classical, like the theory of admissible covers, of J.Harris and D.Mumford ([HM82]), and the theory of limit linear series, of D.Eisenbud and J.Harris ([EH86]). Although these techniques have been successfully applied by their creators to solve important problems, and they have been further studied by others ([B99], [EM02], [O06] for example), several open questions, some considered in the present paper, remain open. Our method, applied also in [C08], is different as it departs from the compactified Picard scheme and does not use degeneration techniques.

We proceed in analogy with the classical theory of Riemann surfaces. Our first result is Theorem 2.2.1, generalizing a theorem of Riemann, computing $h^0(X, L)$ for a balanced line bundle L of large degree on a semistable curve X. Although this theorem fails on infinitely many components of the Picard scheme of a reducible curve (see Example 2.2.3), we prove that, quite pleasingly, it does hold for every balanced line bundle, that is for every element of the compactified Picard scheme of X.

We then turn to study the theorem of Clifford. The situation is much more complex, as this theorem turns out to fail, even for balanced line bundles, in certain situations. Nonetheless, we prove that Clifford's theorem does hold in several cases. Namely, it holds for all degrees on curves with two components (Theorem 3.2.1). Also it holds for all stable curves if the degree is 0 or $2g-2$ (Theorems 3.3.1 and 4.1.6). Finally, it holds for degree at most 4, for all stable curves free from separating nodes (Theorem 4.2.8). Some counterexamples are exhibited to show that the result is sharp: the Clifford inequality fails for all positive degrees for curves with separating nodes; furthermore if $d \geq 5$ then it fails even for curves free from separating nodes (see Example 4.3.6).

The last section is devoted to applications. For curves with two components the Clifford's theorem is valid, it is thus interesting to study their (suitably defined) Clifford's index and its connection with the gonality; we do that in Proposition 5.1.1, stating that a curve is weakly hyperelliptic (i.e. it admits a balanced g^1_2) if and only if its Clifford index is 0. Next, we focus on weakly hyperelliptic curves, give a combinatorial characterization of them (Theorem 5.2.3) and use it to describe the combinatorics of hyperelliptic curves (Proposition 5.2.5). We conclude the paper with a classification of g^2_5's on two-component curves of genus 6 (Theorem 5.3.2).

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1.0.1. *Conventions.* We work over any algebraically closed field. The following notation and terminology will be used thoughout the paper. The word "curve" stands for reduced projective scheme of pure dimension one. X is a connected curve, having at most nodes as singularities. g is the arithmetic genus of X. The irreducible component decomposition of X is written $X = \cup_{i=1}^{\gamma} C_i$, and g_i is the arithmetic genus of C_i. We shall usually denote by Z a (complete, reduced, of pure dimension one) subcurve of X, by g_Z its arithmetic genus, and by $Z^c = \overline{X \smallsetminus Z}$ its complementary curve.

Given a line bundle $L \in \operatorname{Pic} X$ we denote by L_Z its restriction to a subcurve Z of X.

Given two subcurves Z, Z' of X with no components in common, we shall denote

$$Z \cdot Z' := \# Z \cap Z' \quad \text{and} \quad \delta_Z := Z \cdot Z^c = \# Z \cap Z^c. \tag{1}$$

The formula $g = g_Z + g_{Z^c} + \delta_Z - 1$ will be used several times.

Whenever we shall decompose a curve as a union of subcurves, e.g. $X = Z \cup Y$, it will always be understood that Z and Y have no components in common.

$\underline{d} = (d_1, \ldots, d_\gamma)$ will always be an element of $\mathbb{Z}^\gamma$, and $|\underline{d}| = \sum_1^\gamma d_i$. By $\underline{d} \leq 0$ (resp. $\underline{d} \geq 0$) we mean that $d_i \leq 0$ (resp. $d_i \geq 0$) for every i. We denote by $\operatorname{Pic}^{\underline{d}} X$, the set of line bundles L on X having multidegree $d_i = \deg_{C_i} L$ for $i = 1 \ldots \gamma$, and, for any integer $r \geq 0$ we set $W^r_{\underline{d}}(X) := \{L \in \operatorname{Pic}^{\underline{d}} X : h^0(L) \geq r + 1\}$.

1.1. **Gluing global sections.** In this subsection, we collect several technical lemmas needed in the sequel.

1.1.1. Let $\nu : Y \to X$ be some partial (possibly total) normalization of X; consider the (surjective) morphism $\nu^* : \operatorname{Pic} X \to \operatorname{Pic} Y$. For every $M \in \operatorname{Pic} Y$ we will denote the fiber of ν^* over M as follows

$$F_M(X) := \{L \in \operatorname{Pic} X : \nu^* L = M\}. \tag{2}$$

Let δ be the number of nodes normalized by $\nu : Y \to X$. For each of such node, n_i, let $\{p_i, q_i\} = \nu^{-1}(n_i)$ be its branches. We represent the above data by the self explanatory notation

$$Y \longrightarrow X = Y/_{\{p_i = q_i, \quad i=1,\ldots,\delta\}}. \tag{3}$$

Fix $M \in \operatorname{Pic} Y$ such that $h^0(Y, M) \neq 0$. Pick $L \in F_M(X)$; then (cf. [C07] 2.1.1)

$$h^0(Y, M) - \delta \leq h^0(X, L) \leq h^0(Y, M). \tag{4}$$

To study when $h^0(X, L) = h^0(Y, M)$ we introduce a convenient notation.

Definition 1.1.2. Let Y be a curve, $M \in \operatorname{Pic} Y$ and p, q nonsingular points of Y. We say that p and q are a neutral pair of M, and write $p \sim_M q$, if

$$h^0(Y, M - p) = h^0(Y, M - q) = h^0(Y, M - p - q). \tag{5}$$

Remark 1.1.3. Notation as in 1.1.2.

(A) The relation $p \sim_M q$ is an equivalence relation.
(B) If p and q lie in different connected components of Y, $p \sim_M q$ if and only if p and q are base points of M.
(C) $p \sim_{\mathcal{O}_Y} q$ if and only if p and q lie in the same connected component of Y.
(D) If M is very ample, then M has no neutral pair .

Lemma 1.1.4. *Let $Y = Z_1 \coprod Z_2/_{\{p_i = q_i, \ i=1,\ldots,\beta\}}$, where Z_1 and Z_2 are two nodal curves, and $p_1, \ldots, p_\beta$ (respectively $q_1, \ldots, q_\beta$) smooth points of Z_1 (resp. of Z_2). Let $M \in \operatorname{Pic} Y$ and let $p \in Z_1$, $q \in Z_2$ be smooth points of Y. If $p \sim_M q$ then p is a base point of $M_{Z_1}(-\sum_{i=1}^\beta p_i)$ (and q is a base point of $M_{Z_2}(-\sum_{i=1}^\beta q_i)$).*

Proof. Suppose that p is not a base point of $M_{Z_1}(-\sum_{i=1}^\beta p_i)$. Then there exists $s_1 \in H^0(Z_1, M_{Z_1}(-\sum_{i=1}^\beta p_i))$ such that $s_1(p) \neq 0$. Since s_1 vanishes at p_i for $i \leq \beta$, s_1 can be glued to the zero section in $H^0(Z_2, M_{Z_2})$, to give a section $s \in H^0(Y, M)$. By construction, $s(p) \neq 0$ and $s(q) = 0$. Therefore $p \not\sim_M q$. ■

The next Lemma follows trivially from Lemmas 2.2.3 and 2.2.4 in [C07].

Lemma 1.1.5. *Let Y be a nodal curve, p and q two nonsingular points of Y and $Y \to X = Y/_{\{p=q\}}$. Let $M \in \operatorname{Pic} Y$ be such that $h^0(Y, M) \neq 0$.*

There exists $L \in F_M(X)$ such that $h^0(X, L) = h^0(Y, M)$ if and only if $p \sim_M q$. If Y is connected, such an L is unique (if it exists) if and only if p and q are not base points for M.

Lemma 1.1.6. *Let $Y = Z_1 \coprod Z_2 \to X = Y/_{\{p_i=q_i,\ i=1,\dots,\delta\}}$, where $p_1,\dots,p_\delta$ (respectively $q_1,\dots,q_\delta$) are non singular points of Z_1 (resp. of Z_2). Let $M = (M_1, M_2) \in \operatorname{Pic} Z_1 \times \operatorname{Pic} Z_2 = \operatorname{Pic} Y$; assume $h^0(Y,M) \geq 2$, and $p_i \not\sim_M q_i$ $\forall i$. Then there exists $L \in F_M(X)$ such that $h^0(X,L) = h^0(Y,M) - 1$ if and only if*

$$p_i \sim_{M_1} p_j \quad and \quad q_i \sim_{M_2} q_j, \quad \forall i,j.$$

Proof. If $\delta = 1$ we have $F_M(X) = \{L\}$ and our assumption $p_1 \not\sim_M q_1$ implies, by Lemma 1.1.5, that $h^0(X,L) = h^0(Y,M) - 1$. From now on we let $\delta \geq 2$. Assume first $\delta = 2$. Denote $Y' = Y/_{\{p_1=q_1\}}$, and let $M' \in \operatorname{Pic} Y'$ be the (unique) line bundle corresponding to M. As we just said, Lemma 1.1.5 yields

$$h^0(Y', M') = h^0(Y,M) - 1.$$

Suppose $p_2 \not\sim_{M_1} p_1$, Then there is $s_1 \in H^0(Z_1, M_1)$ vanishing at p_1 but not at p_2. Hence p_2 is not a base point of $M_1(-p_1)$. By Lemma 1.1.4 we have $p_2 \not\sim_{M'} q_2$,, hence by Lemma 1.1.5, for every $L \in F_{M'}(X)$ we have $h^0(X,L) \leq h^0(Y',M') - 1 = h^0(Y,M) - 2$.

Conversely, assume $p_2 \sim_{M_1} p_1$ and $q_2 \sim_{M_2} q_1$. We claim that $p_2 \sim_{M'} q_2$. Indeed, pick $s \in H^0(Y', M')$ such that $s(p_2) = 0$. Call s_i the restriction of s to Z_i. Then $s_1 \in H^0(Z_1, M_1)$, hence $s_1(p_1) = 0$ by hypothesis. Therefore $s_2(q_1) = 0$. Finally, as $q_2 \sim_{M_2} q_1$, we get $s_2(q_2) = 0$, hence $s(q_2) = 0$. So $p_2 \sim_{M'} q_2$.

By Lemma 1.1.5 this implies that there exists $L \in F_{M'}(X)$ such that $h^0(X,L) = h^0(Y',M') = h^0(Y,M) - 1$, so we are done.

If $\delta \geq 3$, we just apply the previous argument by replacing p_2, q_2 with p_i, q_i, $i \geq 3$, and use Remark 1.1.3 (A). ■

Fact 1.1.7. Let X be connected, and assume $\underline{d} = \underline{0} = (0,\dots,0)$. Then for every $L \in \operatorname{Pic}^{\underline{0}} X$ we have $h^0(X,L) \leq 1$ and equality holds if and only if $L = \mathcal{O}_X$ (Corollary 2.2.5 of [C07]).

The following easy observation will be applied several times.

Remark 1.1.8. *Let $X = V \cup Z$ and $L \in \operatorname{Pic}^{\underline{d}} X$; assume that $\underline{d}_Z = (0,\dots,0)$. Then $h^0(X,L) \leq h^0(V, L_V)$.*

Indeed, let $Z = Z_1 \coprod \dots \coprod Z_c$ be the connected component decomposition of Z. Then, by Fact 1.1.7, $h^0(Z_i, L_{Z_i}) \leq 1$ and equality holds if and only if $L_{Z_i} = \mathcal{O}_{Z_i}$, in which case L_{Z_i} has no base point. Set $X_1 = V \cup Z_1 \subset X$; if $h^0(Z_1, L_{Z_1}) = 0$ then, obviously, $h^0(X_1, L_{X_1}) \leq h^0(V, L_V)$. If instead $L_{Z_1} = \mathcal{O}_{Z_1}$, by Lemma 1.1.5 applied to X_1 we obtain $h^0(X_1, L_{X_1}) \leq h^0(V, L_V) + 1 - 1 = h^0(V, L_V)$. Iterating, we are done.

Recall the notational conventions of 1.0.1.

Lemma 1.1.9. *Let $X = C \cup Z$ with C irreducible, set $\delta_C = C \cdot Z$. Let $L \in \operatorname{Pic} X$ be such that $degL_C = 2g_C + e_C$ for some $e_C \geq 0$. Then*

(i) $h^0(X,L) \leq h^0(C, L_C) + h^0(Z, L_Z) - \min\{\delta_C, e_C + 1\}$.

(ii) If $e_C \geq \delta_C - 1$ then $h^0(X,L) = h^0(C, L_C) + h^0(Z, L_Z) - \delta_C$.

(iii) If $e_C \leq \delta_C - 2$, equality holds in (i) for at most one L.

Proof. We simplify the notation setting $\delta = \delta_C$. Let $X_0 := C \coprod Z$ and $\nu_0 : X_0 \to X$ be the natural map (the normalization of X at $C \cap Z$). Call $M_0 = (L_C, L_Z) \in \operatorname{Pic} X_0 = \operatorname{Pic} C \times \operatorname{Pic} Z$. We can factor ν_0 by normalizing one node in $C \cap Z$ at the time, as follows. Denote

$$\nu_0 : X_0 \xrightarrow{\nu_1^0} X_1 \xrightarrow{\nu_2^1} \dots \longrightarrow X_{\delta-1} \xrightarrow{\nu_\delta^{\delta-1}} X_\delta = X,$$

so that

$$\nu_{i+1}^i : X_i \longrightarrow X_i/_{\{p_i=q_i\}} = X_{i+1}$$

is the normalization of exactly one node of X_{i+1}, whose branches p_i, q_i satisfy $p_i \in C$ and $q_i \in Z$. For all $i < \delta$, denote $\nu_i : X_i \longrightarrow X$ the composition, and $M_i := \nu_i^* L$. We have, of course,

$$h^0(X, L) \leq h^0(X_i, M_i). \tag{6}$$

Notice that $h^0(X_0, M_0) = h^0(C, L_C) + h^0(Z, L_Z)$.

We claim that, for every $e \leq \min\{\delta - 1, e_C\}$, we have

$$h^0(X_{e+1}, M_{e+1}) = h^0(C, L_C) + h^0(Z, L_Z) - e - 1. \tag{7}$$

By induction on e. If $e = 0$, then $\deg L_C \geq 2g_C$, therefore L_C has no base points. By Lemma 1.1.5 we obtain

$$h^0(X_1, M_1) = h^0(X_0, M_0) - 1 = h^0(C, L_C) + h^0(Z, L_Z) - 1.$$

Assume, as induction hypothesis, that $h^0(X_e, M_e) = h^0(C, L_C) + h^0(Z, L_Z) - e$. Now

$$\deg L_C(-\sum_{j=1}^{e} p_j) = \deg L_C - e \geq 2g_C,$$

therefore $L_C(-\sum_{j=1}^{e} p_j)$ does not have base points; in particular, p_{e+1} is not a base point. By Lemma 1.1.4 we have $p_{e+1} \not\sim_{M_e} q_{e+1}$. By Lemma 1.1.5, this implies

$$h^0(X_{e+1}, M_{e+1}) = h^0(X_e, M_e) - 1 = h^0(C, L_C) + h^0(Z, L_Z) - e - 1$$

proving (7), which, combined with (6), proves (i).

From (7) we also immediately derive (ii).

Finally, for (iii) it suffices to apply the uniqueness part of Lemma 1.1.5. ■

1.2. **Clifford index of a line bundle.** The Clifford index of a line bundle on a curve X is the number $\operatorname{Cliff} L := \deg L - 2h^0(X, L) + 2$. If X is irreducible and $0 \leq \deg L \leq 2g$, then $\operatorname{Cliff} L \geq 0$, by Clifford's theorem; indeed the extension to irreducible nodal of the classical Clifford's theorem for smooth curve is well known (and easy to prove by induction on the genus). Notice also that if $\operatorname{Cliff} L = 0$ then L has no base points, and if $\operatorname{Cliff} L = 1$ then L has at most one base point.

The next Lemma relates $\operatorname{Cliff} L$ to the equivalence $\sim_L$ defined in Definition 1.1.2.

Lemma 1.2.1. *Let C be an irreducible curve of genus g; fix $L \in \operatorname{Pic}^d C$ with $h^0(L) \geq 2$ and $d \leq 2g$. Let E be a set of nonsingular points of C such that $p \sim_L q$ for all $p, q \in E$. Then $\#E \leq \operatorname{Cliff} L + 2$.*

Proof. Let $p_1, \ldots, p_e \in E$; for every $i = 1, \ldots, e$ we have

$$1 \leq h^0(C, L - p_i) = h^0(C, L - \sum_{j=1}^{e} p_j) \leq \frac{d - e}{2} + 1$$

(by Clifford's theorem). On the other hand $h^0(C, L) = d/2 + 1 - \operatorname{Cliff} L/2$, hence

$$h^0(C, L - p_i) \geq \frac{d - \operatorname{Cliff} L}{2}.$$

Therefore

$$\frac{\operatorname{Cliff} L - d}{2} \geq \frac{e - d}{2} - 1 \;\Rightarrow\; \operatorname{Cliff} L + 2 \geq e.$$

■

Corollary 1.2.2. *Let $X = (C_1 \coprod C_2)/_{\{p_i = q_i,\ \ i=1,\ldots,\delta\}}$, with C_1 and C_2 irreducible, and $p_1, \ldots, p_\delta$ (resp. $q_1, \ldots, q_\delta$) nonsingular points of C_1 (resp. of C_2). Pick $L_1 \in \operatorname{Pic} C_1$ globally generated, such that $h^0(C_1, L_1) \geq 2$ and $\operatorname{Cliff} L_1 + 2 < \delta$. Then for any $L_2 \in \operatorname{Pic} C_2$ and any $L \in F_{(L_1, L_2)}(X)$ we have $h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_2, L_2) - 2$.*

Proof. Since $\delta > \operatorname{Cliff} L_1 + 2$, Lemma 1.2.1 yields that there exists at least a pair p_i, p_j such that $p_i \not\sim_{L_1} p_j$. As L_1 is globally generated, by Remark 1.1.3(B) we have $p_i \not\sim_L q_i$ for any L as above; hence Lemma 1.1.6 applies, giving the statement. ∎

In what follows we shall frequently use, without mentioning it, the obvious fact that $\operatorname{Cliff} L$ and $\deg L$ have the same parity.

Proposition 1.2.3. *Let $X = C_1 \cup C_2$ with C_i irreducible of genus g_i. Assume $\delta := C_1 \cdot C_2 \geq 2$. Let $L \in \operatorname{Pic}^{\underline{d}} X$, set $L_i = L_{C_i}$, $d_i = \deg_{C_i} L_i$ and assume $0 \leq d_i \leq 2g_i$ for $i = 1, 2$.*

(i) If $\operatorname{Cliff} L = 0$ then $\operatorname{Cliff} L_1 = \operatorname{Cliff} L_2 = 0$; moreover, if $\underline{d} \neq \underline{0}$ then $\delta = 2$.
(ii) If $\operatorname{Cliff} L = 1$ we may assume d_1 odd and d_2 even. Then $\operatorname{Cliff} L_1 = 1$ and $\operatorname{Cliff} L_2 = 0$. Moreover, if $d_1 \geq 3$, then $\delta \leq 3$; if $d_2 \geq 2$ then $\delta = 2$.
(iii) If $0 \leq \operatorname{Cliff} L \leq 1$, then

$$h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_2, L_2) - 1 \leq d/2 + 1.$$

Proof. Denote $l = h^0(X, L)$ and $l_i = h^0(C_i, L_i)$. Let $p_1, \ldots, p_\delta \in C_1$ and $q_1, \ldots, q_\delta \in C_2$ be the points corresponding to the nodes of X, so that

$$X = (C_1 \coprod C_2)/_{\{p_i = q_i, \quad i=1,\ldots,\delta\}}.$$

Now, as $l \leq l_1 + l_2$ we always have

$$\operatorname{Cliff} L = d - 2l + 2 \geq d - 2l_1 - 2l_2 + 2 = \operatorname{Cliff} L_1 + \operatorname{Cliff} L_2 - 2. \tag{8}$$

Moreover, if either L_1 does not have a base point at some p_i, or L_2 does not have a base point at some q_i, we have $l \leq l_1 + l_2 - 1$, by Lemma 1.1.5 . Therefore

$$\operatorname{Cliff} L = d - 2l + 2 \geq d - 2l_1 - 2l_2 + 2 + 2 = \operatorname{Cliff} L_1 + \operatorname{Cliff} L_2. \tag{9}$$

Recall that if $\operatorname{Cliff} L_i \leq 1$ then L_i has at most one base point. Therefore, as $\delta \geq 2$, (9) applies if either $\operatorname{Cliff} L_1 \leq 1$ or $\operatorname{Cliff} L_2 \leq 1$.

Assume $\operatorname{Cliff} L = 0$. Then (8) yields $\operatorname{Cliff} L_i \leq 2$ for $i = 1, 2$ (as $\operatorname{Cliff} L_i \geq 0$ by Clifford's theorem for irreducible curves). If $\operatorname{Cliff} L_1 = 0$ we can apply (9), obtaining $\operatorname{Cliff} L_2 = 0$. Moreover we have equality occurring in (9), hence $l = l_1 + l_2 - 1$. By Lemma 1.1.6 we obtain that $p_i \sim_{L_1} p_j$ and $q_i \sim_{L_2} q_j$ for all i, j. If $\underline{d} \neq \underline{0}$ and $\delta \geq 3$, this is impossible by Lemma 1.2.1. We conclude that $\delta = 2$.

By switching roles between L_1 and L_2 this argument together with (8) shows that $\operatorname{Cliff} L = 0$ implies $\operatorname{Cliff} L_i \leq 1$ for $i = 1, 2$. If $\operatorname{Cliff} L_1 = 1$ applying (9) gives $0 \geq 1 + \operatorname{Cliff} L_2$, which is impossible. (i) is proved.

Now assume $\operatorname{Cliff} L = 1$; (8) yields $\operatorname{Cliff} L_1 + \operatorname{Cliff} L_2 \leq 3$. If $\operatorname{Cliff} L_1 = 1$ then (9) applies; we get $1 \geq 1 + \operatorname{Cliff} L_2$, hence $\operatorname{Cliff} L_2 = 0$. Similarly, if $\operatorname{Cliff} L_2 = 0$ by (9) we get $\operatorname{Cliff} L_1 = 1$. We thus have that $\operatorname{Cliff} L_1 = 1$ if and only if $\operatorname{Cliff} L_2 = 0$. As d_1 is odd, the only remaining case is $\operatorname{Cliff} L_1 = 3$; this would imply $\operatorname{Cliff} L_2 = 0$ which implies $\operatorname{Cliff} L_1 = 1$, a contradiction. Therefore the case $\operatorname{Cliff} L_1 = 3$ does not occur. In a similar way we see that the case $\operatorname{Cliff} L_2 = 2$ cannot occur (it would imply $\operatorname{Cliff} L_1 = 1$ which implies $\operatorname{Cliff} L_2 = 0$).

Finally, equality holds in (9), so that $l = l_1 + l_2 - 1$. Hence $p_i \sim_{L_1} p_j$ and $q_i \sim_{L_2} q_j$ for all i, j (by Lemma 1.1.6 as before). Now, if either $d_1 \geq 3$ and $\delta \geq 4$, or if $d_2 \geq 2$ and $\delta \geq 3$, this is impossible by Lemma 1.2.1. (ii) is proved.

Part (iii) follows from the previous ones, observing that in both cases L_2 has no base points. Therefore by Lemma 1.1.5 we have $l \leq l_1 + l_2 - 1$. Finally, if $\operatorname{Cliff} L = 0$ we have $l_1 + l_2 - 1 = d_1/2 + 1 + d_2/2 + 1 - 1 = d/2 + 1$. If $\operatorname{Cliff} L = 1$ we have $l_1 + l_2 - 1 = (d_1 + 1)/2 + d_2/2 + 1 - 1 = d/2 + 1/2$; so we are done. ∎

2. Riemann's theorem for semistable curves

The well known Riemann's theorem for a smooth curve C of genus g states the following: if $d \geq 2g-1$ and $L \in \operatorname{Pic}^d C$, then $h^0(C,L) = d-g+1$. More generally, using the normalization and induction on the number of nodes, it is easy to prove the following:

Fact 2.0.4. Let X be a nodal irreducible curve (of genus g) and $L \in \operatorname{Pic}^d X$. Then
(1) If $d \geq 2g-1$ then $h^0(X,L) = d-g+1$.
(2) If $d \geq 2g$ then L is free from base points.
(Part (1) follows from Riemann-Roch and Serre duality, (2) follows from (1)).

By contrast, if X is reducible, Riemann's theorem trivially fails. In fact, for every fixed $d \geq 2g-1$ there exist infinitely many multidegrees $\underline{d}$, with $|\underline{d}| = d$, such that for any $L \in \operatorname{Pic}^{\underline{d}} X$ we have $h^0(X,L) > d-g+1$ (see Example 2.2.3).

On the other hand, it is well known that, for every d, there exists a well defined finite set of multidegrees, of total degree d, which appear as the multidegrees of all line bundles parametrized by the compactified Picard variety of a stable curve X. More precisely, for any stable curve X we shall denote by $\overline{P^d_X}$ the compactified Picard scheme constructed (independently) in [OS79], [S94], [C94], [P96] (known to be all isomorphic by [A04] and [P96]). Recall that $\overline{P^d_X}$ is a reduced scheme of pure dimension g, which appears as the specialization of the degree-d Picard varieties of smooth curves specializing to X. There are several modular descriptions of $\overline{P^d_X}$; the one we shall use interprets its points as equivalence classes of balanced line bundles on curves stably equivalent to X.

The main result of this section, Theorem 2.2.1, states that if L is a line bundle on a semistable curve X, having degree at least $2g-1$, and balanced multidegree, then, just as for smooth curves, we have $h^0(X,L) = d-g+1$. Therefore, if X is stable, every line bundle parametrized by the compactified Picard scheme $\overline{P^d_X}$ satisfies Riemann's theorem.

2.1. **Balanced line bundles.**

Let X be fixed. For every subcurve $Z \subset X$ with $\delta_Z := Z \cdot Z^c$, we set

$$w_Z := \deg_Z \omega_X = 2g_Z - 2 + \delta_Z \quad \text{and} \quad w := w_X = 2g-2. \tag{10}$$

Recall that a (nodal connected) curve X of genus $g \geq 2$ is *stable* if for every subcurve $Z \subset X$ we have $0 < w_Z < w$. X is *semistable* if for every $Z \subset X$ we have

$$0 \leq w_Z \leq w, \tag{11}$$

and $w_Z = 0$ if and only if Z is a union of exceptional components of X (a component $E \subset X$ is called exceptional if $E \cong \mathbb{P}^1$ and if $\delta_E = 2$).

We say that a semistable curve X is *stably equivalent* to a stable curve $\overline{X}$ if $\overline{X}$ is the curve obtained from X by contracting all of its exceptional components. $\overline{X}$ is called the *stabilization* of X.

2.1.1. Let $\underline{d} \in \mathbb{Z}^\gamma$ with $d = |\underline{d}|$; also fix $g \geq 2$. Assume that X is stable. We say that $\underline{d}$ is *balanced* if for every (connected) subcurve $Z \subset X$ we have

$$d\frac{w_Z}{w} - \frac{\delta_Z}{2} \leq d_Z \leq d\frac{w_Z}{w} + \frac{\delta_Z}{2}. \tag{12}$$

More generally, if X is semistable, we say that $\underline{d}$ is balanced if (12) holds, and if for every exceptional component E of X we have $d_E = 1$ (note that if a semistable curve admits some balanced multidegree, then it is quasistable, i.e. two exceptioanl components do not intersect). Set

$$B_d(X) := \{\underline{d} : |\underline{d}| = d,\ \underline{d} \text{ is balanced}\}. \tag{13}$$

A line bundle on a semistable curve is balanced if its multidegree is balanced.

Example 2.1.2. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$ and $1 \leq g_1 \leq g_2$. Pick $d = 2$.

$$B_2(X) = \begin{cases} \{(0,2)\} & \text{if } g_1 < (g+1)/4 \\ \{(0,2);(1,1)\} & \text{if } g_1 = (g+1)/4 \\ \{(1,1)\} & \text{if } g_1 > (g+1)/4 \end{cases}$$

The terminology "balanced" was introduced in [C94] to indicate that balanced multidegrees are closely related to the topological characters of the curve. Indeed, the balanced multidegrees of total degree $d \in \mathbb{Z}$ are as close as they can be to the multidegree $\frac{d}{2g-2}\,\underline{\deg}\,\omega_X$. The word balanced is sometimes replaced by the word "semistable". As we mentioned at the beginning of the section, if X is stable its compactified Picard scheme parametrizes equivalence classes of balanced line bundles on semistable curves having X as stabilization. If X is semistable, its compactified Picard scheme turns out to coincide to the compactified Picard scheme of its stabilization. In the present paper we do not need to be more precise about this point; see loc. cit for details.

2.2. **Positivity properties of balanced line bundles.** We denote

$$X_{\text{sep}} := \{n \in X_{\text{sing}} : n \text{ is a separating node of } X\} \subset X. \tag{14}$$

Theorem 2.2.1 (Balanced Riemann)**.** *Let X be a semistable curve of genus $g \geq 2$, d an integer and $\underline{d} \in B_d(X)$. Let $L \in \operatorname{Pic}^{\underline{d}} X$.*

(i) If $d \geq 2g-1$, then $h^0(X,L) = d - g + 1$.
(ii) If $d \geq 2g$ and $X_{sep} = \emptyset$, then L has no base points.
(iii) If $d \geq 5(g-1)$, then L has no base points.

Part (i) may fail if $\underline{d}$ is not balanced; see Example 2.2.3. Part (ii) may fail if $X_{\text{sep}} \neq \emptyset$; see Example 2.2.4.

Proof. Let $Z \subsetneq X$ be a connected subcurve. We claim that, if $d \geq 2g-1$, we have

$$d_Z \geq 2g_Z - 1 \tag{15}$$

and, if $d \geq 2g$ and $X_{\text{sep}} = \emptyset$, we have

$$d_Z \geq 2g_Z. \tag{16}$$

To prove this, set $d = 2g - 2 + a = w + a$ with $a > 0$. As $\underline{d}$ is balanced, we have

$$d_Z \geq d\frac{w_Z}{w} - \frac{\delta_Z}{2} = 2g_Z - 2 + \frac{\delta_Z}{2} + a\frac{w_Z}{w}.$$

Now, $\delta_Z \geq 1$ and $w_Z \geq 0$ (cf. (11)). Therefore the above inequality yields $d_Z \geq 2g_Z - 1$, as claimed in (15).

To prove (16), assume $X_{\text{sep}} = \emptyset$. Then $\delta_Z \geq 2$, so the previous inequality yields $d_Z \geq 2g_Z$, unless $w_Z = 0$, i.e. unless Z is a chain of exceptional components (recall that X is semistable). If that is the case, $d_Z = 1$ and $g_Z = 0$. So we have $d_Z = 2g_Z + 1 > 2g_Z$. (16) is proved.

Now, part (i) of the Theorem follows from the next Lemma 2.2.2.

We shall apply Lemma 2.2.2 also for part (ii). If $d_Z \geq 2g_Z$ for every Z, then for any nonsingular point $p \in X$ we obviously have $\deg_Z L(-p) \geq 2g_Z - 1$, hence Lemma 2.2.2 applies to $L(-p)$, yielding $h^0(X, L(-p)) = h^0(X,L) - 1$. Now let $n \in X_{\text{sing}}$. Let $\nu : Y \to X$ be the normalization of X at n, $M := \nu^* L$ and $\nu^{-1}(n) = \{q_1, q_2\}$. To prove that L has a section not vanishing at n it suffices to prove that

$$h^0(Y, M(-q_1 - q_2)) = h^0(Y, M) - 2. \tag{17}$$

Let $Z' \subset Y$ be a connected subcurve, and $Z := \nu(Z')$. Then

$$\deg_{Z'} M = \deg_Z L \geq 2g_Z,$$

also $g_Z \geq g_{Z'}$ and strict inequality holds if and only if both q_1 and q_2 lie on Z', in which case $g_Z = g_{Z'} + 1$. Therefore

$$deg_{Z'} M(-q_1 - q_2) \geq \begin{cases} 2g_Z - 2 = 2g_{Z'}, & \text{if } q_1, q_2 \in Z' \\ \\ 2g_Z - 1 \geq 2g_{Z'} - 1, & \text{otherwise.} \end{cases}$$

We can thus apply Lemma 2.2.2, proving (17) as follows:

$$h^0(Y, M(-q_1 - q_2)) = \deg M - 2 - g_Y + 1 = h^0(Y, M) - 2.$$

By the same argument, to prove (iii) it suffices to show that $d_Z \geq 2g_Z$ for every $Z \subset X$. Now, $d \geq 5(g-1)$ implies $d \geq 2g$, so by the previous parts it suffices to consider subcurves Z having $\delta_Z = 1$. Let Z be such a subcurve of X; note that $g_Z \geq 1$ (X is semistable) hence $w_Z = 2g_Z - 2 + \delta_Z \geq 2 - 2 + 1 = 1$. As $\underline{d}$ is balanced, and $d \geq 2(g-1) + 3(g-1) = w + 3(g-1)$, we have

$$d_Z \geq \frac{dw_Z}{w} - \frac{1}{2} \geq w_Z + \frac{3(g-1)w_Z}{2(g-1)} - \frac{1}{2} = 2g_Z - \frac{3}{2} + \frac{3w_Z}{2} \geq 2g_Z.$$

Hence we are done. ∎

Lemma 2.2.2. *Let Y be a (possibly disconnected) curve of genus g and $L \in \operatorname{Pic}^d Y$. If $\deg_Z L \geq 2g_Z - 1$ for every connected subcurve $Z \subseteq Y$, then $h^0(Y, L) = d - g + 1$.*

Proof. Let $X_1, \ldots, X_c$ be the connected components of Y. Then $g = \sum_{i=1}^c g_{X_i} - c + 1$ and $h^0(Y, L) = \sum_{i=1}^c h^0(X_i, L_{X_i})$; therefore it suffices to prove the lemma for a connected curve X of genus g.

We shall use induction on the number of irreducible components of X. The base case, X irreducible, is known (cf. Fact 2.0.4). Assume X reducible. We begin by showing that there exists an irreducible component, C_1, of X such that

$$d_1 \geq 2g_1 + \delta_1 - 1. \tag{18}$$

By contradiction, assume the contrary. Then

$$d = \sum_{i=1}^{\gamma} d_i \leq \sum_{i=1}^{\gamma}(2g_i + \delta_i - 2) = 2\sum_{i=1}^{\gamma} g_i + \sum_{i=1}^{\gamma} \delta_i - 2\gamma.$$

Now, $\sum_{i=1}^{\gamma} \delta_i = 2\delta$ and $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$. Therefore

$$d \leq 2(\sum_{i=1}^{\gamma} g_i + \delta - \gamma) = 2(g-1),$$

contradicting the assumption $d \geq 2g - 1$. This proves (18).

Let us write $X = C_1 \cup Z$ with $Z = C_1^c$. Let $Z = Z_1 \coprod \ldots \coprod Z_c$, with Z_i connected. We use induction and get

$$h^0(Z_i, L_{Z_i}) = d_{Z_i} - g_{Z_i} + 1. \tag{19}$$

Now, by (18) we can apply Lemma 1.1.9(ii) and obtain

$$h^0(X, L) = h^0(C_1, L_1) + h^0(Z, L_Z) - \delta_1 = d - (g_1 + \sum_{i=1}^{c} g_{Z_i}) + c + 1 - \delta_1$$

(using $h^0(C_1, L_1) = d_1 - g_1 + 1$ and (19)). Now $g = g_1 + \sum_{i=1}^c g_{Z_i} + \delta_1 - c$, hence $h^0(X, L) = d - g + \delta_1 - c + c + 1 - \delta_1 = d - g + 1$. ∎

Example 2.2.3. Fix X having $\gamma \geq 2$ components and genus g; let $d \geq 2g-1$. The theorem of Riemann fails for all but finitely many $\underline{d}$ with $|\underline{d}| = d$. To prove that it will be enough to show the following. For every fixed $i \in \{1, \dots, \gamma\}$ there exists m_i such that for every $\underline{d}$ such that $d_i \geq m_i$ and for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have $h^0(X, L) > d - g + 1$.

So, pick $i = 1$, let $m_1 := d + g_1 + \delta_1 + 1$ ($\delta_1 = C_1 \cdot C_1^c$). If $d_1 \geq m_1$ we have

$$d_1 \geq d + g_1 + \delta_1 + 1 \geq 2g - 1 + g_1 + \delta_1 + 1 \geq 2g_1 + g_1 + \delta_1 = 3g_1 + \delta_1 \geq 2g_1 + 1;$$

hence $h^0(C_1, L_1) = d_1 - g_1 + 1$. Now, for any $L \in \operatorname{Pic}^{\underline{d}} X$ such that $d_1 \geq m_1$ (we can adjust the remaining $d_2, \dots, d_\gamma$ however we like so that $|\underline{d}| = d$)

$$h^0(X, L) \geq h^0(C_1, L_1) - \delta_1 = d_1 - g_1 + 1 - \delta_1 \geq d + g_1 + \delta_1 + 1 - g_1 + 1 - \delta_1$$

hence $h^0(X, L) \geq d + 2 > d - g + 1$ as wanted.

Example 2.2.4. If X has a separating node part (ii) of Theorem 2.2.1 may fail. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$. Assume $g_1 = 1$ and $g_2 = g - 1$ and $d = 2g + b$ with $b \geq 0$. Let $\underline{d} = (1, d-1) = (1, 2g + b - 1) = (1, 2g_2 + b + 1)$, if $g \geq b + 3$ one checks that $\underline{d}$ is balanced. Set $L = (\mathcal{O}_{C_1}(p), L_2)$ such that $p \neq C_1 \cap C_2$. Assume for simplicity that L_2 has no base point in $C_1 \cap C_2$. Then

$$h^0(X, L) = h^0(C_1, \mathcal{O}_{C_1}(p)) + h^0(C_2, L_2) - 1 = h^0(C_2, L_2).$$

Now, L has a base point in p, indeed

$$h^0(X, L(-p)) = h^0(C_1, \mathcal{O}_{C_1}) + h^0(C_2, L_2) - 1 = h^0(C_2, L_2).$$

3. Clifford's Theorem for all degrees

In this section we prove the following cases of Clifford's theorem: Theorem 3.2.1, for curves with two components and every balanced multudegree; Theorem 3.3.1 for all curves and all balanced line bundles of degree $2g-2$; Proposition 3.1.1 for all curves and all degrees, provided the hypothesis that the degree be at most twice the genus is "uniformly" satisfied on all irreducible components.

3.1. **Uniform extension.**

Proposition 3.1.1 (Uniform Clifford)**.** *Let X be a connected curve of genus g. Let $\underline{d} = (d_1, \dots, d_\gamma) \in \mathbb{Z}^\gamma$ be such that $0 \leq d_i \leq 2g_i$ for every $i = 1, \dots, \gamma$.*

(i) Then $|\underline{d}| \leq 2g$ and for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have $h^0(X, L) \leq \deg L/2 + 1$.
(ii) If equality holds and $|\underline{d}| \leq 2g - 2$ then L has no nonsingular base points (i.e. if L admits a base point, this point is a node of X).

Proof. As we said in Subsection 1.2 we may assume X reducible. Set $|\underline{d}| = d$.

Let us prove that $d \leq 2g$. We have $d = \sum_{i=1}^{\gamma} d_i \leq \sum_{i=1}^{\gamma} 2g_i$. Let δ be the number of nodes of X that lie in two different irreducible components. Then $g = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$. On the other hand, as X is connected, we have $\delta \geq \gamma - 1$. Therefore $2g - d \geq 2g - 2\sum_{i=1}^{\gamma} g_i = 2(\delta - \gamma + 1) \geq 0$, as claimed.

We continue using induction on the number of irreducible components.

By Remark 3.3.4, we can decompose $X = Z_1 \cup Z_2$ so that the Z_i are connected. We set $l_i := h^0(Z_i, L_{Z_i})$; by the induction assumption, $l_i \leq \frac{d_{Z_i}}{2} + 1$ and if equality holds, L_{Z_i} has no nonsingular base points. We distinguish three cases.

Case 1: $l_i < \frac{d_{Z_i}}{2} + 1$ for both $i = 1, 2$.

If d_{Z_1} and d_{Z_2} are even, then $l_i \leq \frac{d_{Z_i}}{2}$. Hence $h^0(X, L) \leq l_1 + l_2 \leq \frac{d}{2}$.

If d_{Z_1} is even and d_{Z_2} is odd, then $l_1 \leq \frac{d_{Z_1}}{2}$ and $l_2 \leq \frac{d_{Z_2}+1}{2}$. Hence $h^0(X, L) \leq l_1 + l_2 \leq \frac{d+1}{2} < \frac{d}{2} + 1$.

Finally, assume d_{Z_1} and d_{Z_2} odd. Then $l_i \leq \frac{d_{Z_i}+1}{2}$ hence

$$h^0(X, L) \leq l_1 + l_2 \leq \frac{d}{2} + 1.$$

If equality holds we get $l_i = \frac{d_{Z_i}+1}{2}$ for $i = 1, 2$, and $h^0(X, L) = l_1 + l_2$. Therefore $L_{Z_1}$and L_{Z_2} have a base point over every node in $Z_1 \cap Z_2$. This implies that $Z_1 \cdot Z_2 = 1$. Indeed, by induction, the Clifford inequality holds on Z_i, yielding that L_{Z_i} can have at most one base point (indeed, if L_{Z_i} had two base points, p and p', then $h^0(L_{Z_i}(-p-p')) = h^0(L_{Z_i}) = \frac{d_{Z_i}+1}{2} > \frac{d_{Z_i}-2}{2} + 1$).

Let $q_i \in Z_i$ be the branch of the node $n = Z_1 \cap Z_2$. Let $p \in X$ be a point with $p \neq n$, say $p \in Z_1$. If p is a base point for L then it is also a base point for L_{Z_1}, but this is not possible as we just proved that the only base point of L_{Z_1} is q_1.

The proof of (i) and (ii) in Case 1 is complete.

Case 2: $l_1 = \frac{d_{Z_1}}{2} + 1$ and $l_2 < \frac{d_{Z_2}}{2} + 1$.

By induction, L_{Z_1} has no nonsingular base point. Therefore, by Lemma 1.1.5

$$h^0(X, L) \leq l_1 + l_2 - 1 < \frac{d_{Z_1}}{2} + 1 + \frac{d_{Z_2}}{2} + 1 - 1 = \frac{d}{2} + 1.$$

So, in this case strict inequality always holds and we are done.

Case 3: $l_i = \frac{d_{Z_i}}{2} + 1$ for both $i = 1, 2$.

By induction L_{Z_i} is free from nonsingular base points. We get, again by Lemma 1.1.5,

$$h^0(X, L) \leq l_1 + l_2 - 1 = \frac{d_{Z_1}}{2} + 1 + \frac{d_{Z_2}}{2} + 1 - 1 = \frac{d}{2} + 1.$$

Now equality holds if and only if $h^0(X, L) = l_1 + l_2 - 1$. Let $p \in X$ be a nonsingular point, say $p \in Z_1$. As p is not a base point of L_{Z_1}, we have

$$h^0(X, L(-p)) \leq h^0(Z_1, L_{Z_1}(-p)) + l_2 - 1 = l_1 - 1 + l_2 - 1 = h^0(X, L) - 1$$

hence p is not a base point of L, so we are done. ■

Corollary 3.1.2. *Assumptions as in Proposition 3.1.1. Assume* $0 < |\underline{d}| < 2g - 2$. *If there exists* $L \in \operatorname{Pic}^{\underline{d}} X$ *such that* $\operatorname{Cliff} L = 0$, *then for every decomposition* $X = Z_1 \cup Z_2$ *with* Z_1 *connected and* Z_2 *irreducible, we have*

(a) $Z_1 \cdot Z_2 \leq 2$,

(b) If d_{Z_1} *and* d_{Z_2} *are even, then* $\operatorname{Cliff} L_{Z_i} = 0$ *and* $h^0(Z_i, L_{Z_i}(-Z_1 \cap Z_2)) = h^0(Z_i, L_{Z_i}) - 1$, *for* $i = 1, 2$.

(c) If d_{Z_1} *and* d_{Z_2} *are odd, then* $Z_1 \cdot Z_2 = 1$ *and* $\operatorname{Cliff} L_{Z_i}(-Z_1 \cap Z_2) = 0$ *for* $i = 1, 2$.

Proof. We use the proof of Proposition 3.1.1. In Case 1, $\operatorname{Cliff} L = 0$ exactly when the d_{Z_i} are both odd, Z_1 and Z_2 intersect in only one point, and

$$h^0(Z_i, L_{Z_i}) = h^0(Z_i, L_{Z_i}(-q_i)) = \frac{d_{Z_i} + 1}{2} = \frac{d_{Z_i} - 1}{2} + 1.$$

So $\operatorname{Cliff}(L_{Z_i}(-q_i)) = 0$. Observe that we did not use the irreducibility of Z_2.

In Case 2 equality never holds.

In Case 3 we have $\operatorname{Cliff} L = 0$ exactly when the d_{Z_i} are even, $\operatorname{Cliff} L_{Z_i} = 0$ for $i = 1, 2$, and $h^0(X, L) = h^0(Z_1, L_{Z_1}) + h^0(Z_2, L_{Z_2}) - 1$. Notice that by Lemma 1.1.6 this implies that for every pair of points $q, q' \in Z_1 \cap Z_2 \subset Z_2$ we have $q \sim_{L_{Z_2}} q'$ (and similarly for Z_1).

To complete the proof, we need to show that $Z_1 \cdot Z_2 \leq 2$. By contradiction, assume $Z_1 \cdot Z_2 \geq 3$; then a relation $q \sim_{L_{Z_2}} q' \sim_{L_{Z_2}} q''$ holds on Z_2. Observe also that L_{Z_2} has no nonsingular base points, as $\operatorname{Cliff} L_{Z_2} = 0$. Therefore

$$h^0(Z_2, L_{Z_2}(-q - q' - q'')) = h^0(Z_2, L_{Z_2}(-q)) = l_2 - 1 = \frac{d_{Z_2}}{2}.$$

But Z_2 is irreducible, hence Clifford applies to $L_{Z_2}(-q-q'-q'')$, and we get

$$h^0(Z_2, L_{Z_2}(-q-q'-q'')) \leq \frac{d_{Z_2}-3}{2}+1 < \frac{d_{Z_2}}{2},$$

a contradiction. ■

3.2. **Curves with two components.** Clifford's inequality holds for curves with two irreducible components, by the following result.

Theorem 3.2.1. *Let $X = C_1 \cup C_2$ be a semistable curve of genus $g \geq 2$. Let $0 \leq d \leq 2g$ and $\underline{d} \in B_d(X)$. Then for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have*

$$h^0(X, L) \leq d/2 + 1. \tag{20}$$

Addendum 3.2.2. *Let $\epsilon := 1+\max\{d_1-2g_1, d_2-2g_2, 0\}$, and $\beta := \min\{C_1 \cdot C_2, \epsilon\}$. If $C_1 \cdot C_2 \geq 2$, then $h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_2, L_2) - \beta \leq d/2 + 1$.*

Proof. Set $l := h^0(X, L)$, and for $i = 1, 2$, $L_i := L_{C_i}$, $l_i := h^0(C_i, L_i)$. As usual, set $\delta := C_1 \cdot C_2$. By Theorem 2.2.1 we can assume $d \leq 2g-2$. We begin with
Case 0. *If $d_1 < 0$ then (20) holds, with strict inequality if $d \leq 2g-2$.*

As $d_1 < 0$ we have $d_2 > 0$. Since $\underline{d}$ is balanced,

$$d_1 \geq \frac{dw_1}{w} - \frac{\delta}{2} \geq -\frac{\delta}{2} \tag{21}$$

($\frac{dw_1}{w} \geq 0$ as X is semistable). Of course $l_1 = 0$, therefore, denoting by $G_2 \in \operatorname{Div} C_2$ the degree δ divisor cut on C_2 by C_1, a section of L has to vanish on G_2, i.e.

$$h^0(X, L) = h^0(C_2, L_2(-G_2)). \tag{22}$$

Note that $\deg L_2(-G_2) = d_2 - \delta$. If $d_2 - \delta < 0$ we get $h^0(X, L) = 0$ and we are done. If $0 \leq d_2 - \delta \leq 2g_2$ we can use Clifford on C_2 and obtain

$$h^0(C_2, L_2(-G_2)) \leq \frac{d_2-\delta}{2}+1 = \frac{d-d_1-\delta}{2}+1 \leq \frac{d+\delta/2-\delta}{2}+1$$

(using (21)). Combining the above with (22) yields

$$h^0(X, L) \leq \frac{d}{2}+1-\frac{\delta}{4} < \frac{d}{2}+1$$

as stated. Finally, it remains to treat the case $d_2 - \delta \geq 2g_2$, i.e.

$$l = h^0(C_2, L_2(-G_2)) = d_2 - \delta - g_2 + 1.$$

We argue by contradiction, assuming that $l \geq \frac{d}{2}+1$. This is to say, by (22),

$$d_2 - \delta - g_2 + 1 \geq \frac{d}{2}+1,$$

hence (using $d = d_1 + d_2$)

$$\frac{d_2-d_1}{2} - \delta - g_2 \geq 0,$$

equivalently

$$d_2 - d_1 - 2\delta - 2g_2 \geq 0. \tag{23}$$

On the other hand, as $\underline{d}$ is balanced, we have

$$d_2 \leq \frac{dw_2}{w} + \frac{\delta}{2} \quad \text{and} \quad d_1 \geq \frac{dw_1}{w} - \frac{\delta}{2}.$$

Using these two inequalities we get

$$d_2 - d_1 - 2\delta - 2g_2 \leq \frac{dw_2}{w} + \frac{\delta}{2} - \frac{dw_1}{w} + \frac{\delta}{2} - 2\delta - 2g_2 = \frac{d}{w}(w_2 - w_1) - \delta - 2g_2.$$

Now, $w_2 - w_1 = 2g_2 - 2g_1$ and $\frac{d}{w} \le 1$ (as $d \le 2g-2 = w$). We obtain

$$d_2 - d_1 - 2\delta - 2g_2 \le \frac{d}{w}(2g_2 - 2g_1) - \delta - 2g_2 \le -\frac{2dg_1}{w} - \delta < 0$$

contradicting (23). This finishes Case 0.

For the rest of the proof, we can restrict to $d_i \ge 0$ for $i = 1, 2$. By Propositions 3.1.1 and 1.2.3 (iii), we can assume that $d_i \ge 2g_i + 1$ for at least one i, so let $d_1 \ge 2g_1 + 1$. Then $l_1 = d_1 - g_1 + 1$.

Case 1. *If $d_1 \ge 2g_1 + \delta - 1$, then (20) holds, with strict inequality if $d \le 2g - 1$.*

By Lemma 1.1.9(ii),

$$l = l_1 + l_2 - \delta. \tag{24}$$

Subcase 1a. $d_2 \ge 2g_2$. Hence $l_2 = d_2 - g_2 + 1$. Combining with (24) we have

$$l = d_1 - g_1 + 1 + d_2 - g_2 + 1 - \delta = d - (g_1 + g_2 + \delta - 1) + 1 = d - g + 1.$$

Now $d \le 2g$, hence

$$l = d - g + 1 \le d - \frac{d}{2} + 1 = \frac{d}{2} + 1.$$

So we are done. Note that equality holds if and only if $d = 2g$.

Subcase 1b. $d_2 < 2g_2$. By Proposition 3.1.1, $l_2 \le \frac{d_2}{2} + 1$. Set

$$d_1 = 2g_1 + \delta - 1 + a$$

so that $a \ge 0$ and

$$g_1 = \frac{d_1 - \delta + 1 - a}{2}. \tag{25}$$

Using (24) and (25) we get

$$l \le d_1 - g_1 + 1 + \frac{d_2}{2} + 1 - \delta = d_1 - \frac{d_1 - \delta + 1 - a}{2} + 2 + \frac{d_2}{2} - \delta,$$

hence

$$l \le \frac{d}{2} + 1 + \frac{1 - \delta + a}{2}.$$

The subsequent Lemma 3.2.3 yields

$$a \le \begin{cases} \frac{\delta}{2} - 1, & \text{if } \delta \text{ is even} \\[2ex] \frac{\delta - 1}{2} - 1, & \text{if } \delta \text{ is odd} \end{cases}$$

Hence $1 + a \le \frac{\delta}{2}$, so that $1 + a - \delta \le -\frac{\delta}{2} < 0$. We conclude $h^0(X, L) < \frac{d}{2} + 1$ and we are done.

Case 2. *Assume* $2g_1 + 1 \le d_1 < 2g_1 + \delta - 1$.

Set $d_1 = 2g_1 + e_1$ where $1 \le e_1 \le \delta - 2$. Hence

$$g_1 = \frac{d_1 - e_1}{2}. \tag{26}$$

By Lemma 1.1.9 we have

$$l \le l_1 + l_2 - e_1 - 1. \tag{27}$$

If $d_2 \le 2g_2$, then $l_2 \le \frac{d_2}{2} + 1$. Using (26) we have

$$l \le d_1 - g_1 + 1 + \frac{d_2}{2} + 1 - e_1 - 1 = d_1 - \frac{d_1 - e_1}{2} + \frac{d_2}{2} + 1 - e_1 = \frac{d}{2} + 1 - \frac{e_1}{2}.$$

Now $e_1 \ge 1$ hence $l < \frac{d}{2} + 1$ and we are done. Also, strict inequality holds.

If $d_2 \ge 2g_2 + 1$, set $d_2 = 2g_2 + e_2$ with $e_2 \ge 1$. We can also assume $e_2 \le \delta - 1$, otherwise we are done by Case 1 (interchanging C_1 with C_2).

Now the situation is symmetric between C_1 and C_2, so up to switching them we may assume $e_1 \geq e_2$. By Lemma 1.1.9 we have,

$$l \leq l_1 + l_2 - e_1 - 1 = d_1 - g_1 + 1 + d_2 - g_2 + 1 - e_1 - 1.$$

Now, using (26) applied also to C_2

$$l \leq d_1 - \frac{d_1 - e_1}{2} + 1 + d_2 - \frac{d_2 - e_2}{2} + 1 - e_1 - 1 = \frac{d}{2} + 1 + \frac{e_2 - e_1}{2}.$$

As $e_1 \geq e_2$ we conclude $l \leq \frac{d}{2} + 1$. Moreover, equality holds if $e_1 = e_2$ and $l = l_1 + l_2 - e_1 - 1$. ■

Lemma 3.2.3. *Let X be a semistable curve of genus $g \geq 2$, $d \leq 2g - 2$ and $\underline{d} \in B_d(X)$. Let $Z \subset X$ be a subcurve, set $d_Z = 2g_Z + \delta_Z - 1 + a_Z$. Then*

$$a_Z \leq \begin{cases} \frac{\delta_Z}{2} - 1, & \text{if } \delta_Z \text{ is even} \\ \\ \frac{\delta_Z - 1}{2} - 1, & \text{if } \delta_Z \text{ is odd} \end{cases}$$

Proof. We just need to apply (12) and compute, using $d \leq 2g - 2 = w$:

$$d_Z \leq \frac{d w_Z}{w} + \frac{\delta_Z}{2} \leq w_Z + \frac{\delta_Z}{2} = 2g_Z - 2 + \delta_Z + \frac{\delta_Z}{2}.$$

Now the statement follows at once from

$$d_Z = 2g_Z + \delta_Z - 1 + a_Z \leq 2g_Z + \delta_Z - 2 + \frac{\delta_Z}{2}.$$

■

3.3. **Clifford's Theorem in degree $2g - 2$.** The following statement summarizes our results for $d = 2g - 2$.

Theorem 3.3.1. *Let X be a connected curve of genus $g \geq 2$. Let $\underline{d}$ be a multidegree such that $|\underline{d}| = 2g - 2$. Assume that one of the following conditions hold.*

(1) $d_Z \geq 2g_Z - 1$ for every proper subcurve $Z \subsetneq X$.
(2) X is semistable and $\underline{d}$ is balanced.
(3) $0 \leq d_i \leq 2g_i$, for every $i = 1, \ldots, \gamma$.

Then $h^0(X, L) \leq g$ for every $L \in \operatorname{Pic}^{\underline{d}} X$.

Moreover, let $L \in \operatorname{Pic}^{\underline{d}} X$ be such that $h^0(X, L) = g$. If (1) or (2) holds, or if (3) holds with $X_{sep} = \emptyset$, then $L \cong \omega_X$.

Proof. If assumption (1) holds, then the theorem is proved in the subsequent Proposition 3.3.3. Next, (2) implies (1). Indeed,

$$d_Z \geq w_Z \frac{d}{w} - \frac{\delta_Z}{2} = 2g_Z - 2 + \delta_Z - \frac{\delta_Z}{2} = 2g_Z - 2 + \frac{\delta_Z}{2} \geq 2g_Z - \frac{3}{2}.$$

As d_Z is an integer, we obtain $d_Z \geq 2g_Z - 1$. This settles the theorem under hypothesis (2).

If (3) holds, the fact that $h^0(L) \leq g$ is a special case of Proposition 3.1.1.

Now let L be such that $h^0(L) = g$. By Riemann-Roch and Serre duality this is equivalent to

$$h^0(\omega_X \otimes L^{-1}) = 1. \tag{28}$$

Now, $\deg \omega_X \otimes L^{-1} = 0$ and we claim that $\underline{\deg}\, \omega_X \otimes L^{-1} \geq 0$. Indeed, as X is free from separating nodes, for every $i = 1, \ldots, \gamma$ we have $\delta_i \geq 2$. Hence

$$\deg_{C_i} \omega_X \otimes L^{-1} = 2g_i - 2 + \delta_i - d_i \geq 2g_i - d_i \geq 0.$$

Now, by Fact 1.1.7, (28) is possible if and only if $\omega_X \otimes L^{-1} \cong \mathcal{O}_X$, as claimed. ■

Example 3.3.2. The assumption that X_{sep} be empty is indeed necessary in the last part of Theorem 3.3.1, as the present example shows. Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$. Then $2g-2 = 2g_1 + 2g_2 - 2$; assume $g_1 \geq 1$. Let $\underline{d} = (2g_1 - 2, 2g_2)$ and $L = (\omega_{C_1}, L_2)$ for any $L_2 \in \operatorname{Pic}^{2g_2} C_2$. Then, as ω_{C_1} is free from base points, by Lemma 1.1.5 we have

$$h^0(X, L) = h^0(C_1, \omega_{C_1}) + h^0(C_2, L_{C_2}) - 1 = g_1 + (2g_2 - g_2 + 1) - 1 = g.$$

The following is a part of Theorem 3.3.1.

Proposition 3.3.3. *Fix X of genus g and $\underline{d}$ such that $|\underline{d}| = 2g-2$; assume $d_Z \geq 2g_Z - 1$ for every $Z \subsetneq X$. Then for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have*

$$h^0(X, L) \leq g. \tag{29}$$

If equality holds, then $L = \omega_X$.

Proof. We set $l = h^0(X, L)$, $l_i = h^0(C_i, L_i)$ and for any subcurve $Z \subset X$, $l_Z = h^0(Z, L_Z)$. The hypothesis allows us to apply Lemma 2.2.2, getting

$$l_Z = d_Z - g_Z + 1 \tag{30}$$

for every $Z \subsetneq X$.

Step 1. *If there exists i such that $d_i \geq 2g_i + \delta_i - 1$ (in particular, $\underline{d} \neq \underline{\deg}\, \omega_X$), then (29) holds with strict inequality.*

Assume $d_1 \geq 2g_1 + \delta_1 - 1$. We can apply Lemma 1.1.9 to $X = C_1 \cup Z$ where $Z = C_1^c$. Using (30) we obtain

$$l = l_1 + l_Z - \delta_1 = d_1 - g_1 + 1 + d_Z - g_Z + 1 - \delta_1 = d - (g_1 + g_Z + \delta_1 - 1) + 1.$$

Now, $g = g_1 + g_Z + \delta_1 - 1$ hence $l = d - g + 1 = g - 1 < \frac{d}{2} + 1$, as claimed.

Step 2. *If $d_i \leq 2g_i + \delta_i - 2$ for every i, then $\underline{d} = \underline{\deg}\, \omega_X$.*

Set $d_i = 2g_i + e_i$, then

$$\sum_{i=1}^{\gamma} e_i = 2(\delta - \gamma). \tag{31}$$

This is trivial: on the one hand $d = 2g - 2 = \sum_{i=1}^{\gamma}(2g_i + e_i)$. On the other $2g - 2 = 2\sum_{i=1}^{\gamma} g_i + 2\delta - 2\gamma$. So it suffices to compare the two identities.

Now, as $e_i \leq \delta_i - 2$ by assumption, we have

$$2(\delta - \gamma) = \sum_{i=1}^{\gamma} e_i \leq \sum_{i=1}^{\gamma}(\delta_i - 2) = \sum_{i=1}^{\gamma} \delta_i - 2\gamma = 2\delta - 2\gamma$$

therefore equality must hold, which can only happen if $e_i = \delta_i - 2$ for every i. This is of course the same as saying $d_i = \deg_{C_i} \omega_X$, so we are done.

Step 3. *If $d_i \leq 2g_i + \delta_i - 2$ for every i, then the statement holds.*

By Step 2 the hypthesis is equivalent to $\underline{d} = \underline{\deg}\, \omega_X$. By Step 1 this is the only case that remains to be treated. By Remark 3.3.4 we can order the irreducible components of X in such a way that for every $i \neq \gamma$ we have

$$C_i \cap (\cup_{j=i+1}^{\gamma} C_j) \neq \emptyset. \tag{32}$$

Denote $\delta_{i,j} := C_i \cdot C_j$ for every $i \neq j$. Our choice of ordering of the C_i yields $\sum_{j=1}^{i-1} \delta_{i,j} \leq \delta_i - 1$, for all $i < \gamma$. Therefore (as $e_i + 1 = \delta_i - 1$)

$$\min\{e_i + 1, \sum_{j=1}^{i-1} \delta_{i,j}\} = \min\{\delta_i - 1, \sum_{j=1}^{i-1} \delta_{i,j}\} = \sum_{j=1}^{i-1} \delta_{i,j}, \quad \forall i \neq \gamma. \tag{33}$$

Now we shall bound l by gluing one component at the time, starting with gluing C_2 to C_1 and ending with gluing C_γ to $\cup_{i=1}^{\gamma-1} C_i$. At each step we apply Lemma 1.1.9.

So, set $Z_i = \cup_{j=i}^{i} C_j \subset X$. The first gluing (of C_2 to C_1) yields, using (33) and assuming $\gamma \geq 3$ (if $\gamma = 2$ we jump to the last step, gluing $C_\gamma = C_2$ to C_1),

$$h^0(Z_2, L_{Z_2}) \leq l_1 + l_2 - \min\{e_2 + 1, \delta_{1,2}\} = l_1 + l_2 - \delta_{1,2}.$$

More generally, iterating up to the index $i \leq \gamma - 1$, applying Lemma 1.1.9 and (33) at each step, we obtain

$$h^0(Z_i, L_{Z_i}) = l_1 + \ldots + l_i - \delta_{1,2} - \ldots - \sum_{j=1}^{i-1} \delta_{i,j}. \tag{34}$$

The last step is the gluing of C_γ, for which we need

$$\min\{e_\gamma + 1, \delta_\gamma\} = \min\{\delta_\gamma - 1, \delta_\gamma\} = \delta_\gamma - 1; \tag{35}$$

hence

$$l \leq h^0(Z_{\gamma-1}, L_{Z_{\gamma-1}}) + l_\gamma - \min\{e_\gamma + 1, \delta_\gamma\} = h^0(Z_{\gamma-1}, L_{Z_{\gamma-1}}) + l_\gamma - \delta_\gamma + 1.$$

Combining everything we obtain

$$l \leq \sum_{i=1}^{\gamma} l_i - \sum_{i=2}^{\gamma-1} (\sum_{j=1}^{i-1} \delta_{i,j}) - \delta_\gamma + 1 = \sum_{i=1}^{\gamma} l_i - \delta + 1 = d - g + 2 = g$$

$(\sum_{i=1}^{\gamma} l_i = d - \sum_{i=1}^{\gamma} g_i + \gamma = d - g + \delta + 1)$.

This finishes the proof of (29). Observe that in our computation we had equality holding at every step (see (34)) but the last one, when we glued C_γ. At that point, by (35), we are in the situation of Lemma 1.1.9 (iii). We obtain that equality holds for at most one L. Now, if $L = \omega_X$, equality does hold, so this is the only case for which $h^0(X, L) = \frac{d}{2} + 1 = g$. ■

We used the following simple facts, which can be easily proved by induction.

Remark 3.3.4. Let X be a reducible, connected curve.

(i) Then X admits an irreducible component C such that C^c is connected (such a C will be called a *non-disconnecting component*).
(ii) The irreducible components $C_1, \ldots, C_\gamma$ of X can be ordered so that for every $i < \gamma$ there exists $j > i$ such that $C_i \cap C_j \neq \emptyset$.

4. Clifford's Theorem in low degree

4.1. Line bundles of degree at most 0.

4.1.1. Let X be fixed. For any $\underline{d} = (d_1, \ldots, d_\gamma) \in \mathbb{Z}^\gamma$, we denote

$$Z_{\underline{d}}^- := \bigcup_{i: d_i < 0} C_i \subset X. \tag{36}$$

Remark 4.1.2. *Let X be a nodal connected curve, and let $\underline{d}$ be such that $|\underline{d}| < 0$ and $\underline{d} \leq 0$. Then for every $L \in \operatorname{Pic}^{\underline{d}} X$ we have $h^0(X, L) = 0$.*

Indeed $h^0(Z_{\underline{d}}^-, L_{Z_{\underline{d}}^-}) = 0$, of course. Now, for any connected component, Y, of $\overline{X \smallsetminus Z_{\underline{d}}^-}$, we have $\underline{d}_Y = (0, \ldots, 0)$, hence $h^0(Y, L_Y) \leq 1$ with equality if and only if $L_Y = \mathcal{O}_Y$, in which case L_Y has no base points. So the remark follows from Lemma 1.1.5.

Fix $L \in \operatorname{Pic}^{\underline{d}} X$; for every nonzero $s \in H^0(X, L)$ we let Y_s be the subcurve of X where s does not vanish, and W_s its complementary curve:

$$Y_s := \bigcup_{i: s_{|C_i} \neq 0} C_i \subset X \quad \text{and} \quad W_s := \overline{X \smallsetminus Y_s}. \tag{37}$$

Note that $Z_{\underline{d}}^- \subset W_s$ and $\underline{d}_{Y_s} \geq 0$.

Remark 4.1.3. *With the above notation, fix $\underline{d}$ such that $\underline{d} \not\geq 0$, and let $L \in \operatorname{Pic}^{\underline{d}} X$. For every nonzero $s \in H^0(X, L)$ (if it exists) we have $d_{Y_s} \geq \delta_{Y_s}$.*

Indeed $Z^-_{\underline{d}}$ is nonempty, hence W_s is nonempty. Since s vanishes on $W_s \cap Y_s$ the claim follows.

Lemma 4.1.4. *Let X be a semistable curve, $d \leq 0$ and $\underline{d} \in B_d(X)$.*

Then for every $L \in \operatorname{Pic}^{\underline{d}} X$, with $L \neq \mathcal{O}_X$, we have $h^0(X, L) = 0$.

Proof. If $\underline{d} = (0, \dots, 0)$ the statement follows from Fact 1.1.7. We can thus assume $\underline{d} \not\geq 0$. As $\underline{d}$ is balanced, for every subcurve $Z \subset X$ we have

$$d_Z \leq \frac{dw_Z}{w} + \frac{\delta_Z}{2} \leq \frac{\delta_Z}{2}.$$

Hence $d_Z < \delta_Z$. Combining this with Remark 4.1.3, we are done. ∎

4.1.5. By Riemann-Roch and Serre duality, any statement about sections of line bundles of degree $2g - 2$ has a dual statement about sections of line bundles of degree 0. The following is the dual of Theorem 3.3.1.

Theorem 4.1.6 (Clifford for $d = 0$)**.** *Let X be a curve of genus $g \geq 2$. Let $\underline{d}$ be such that $|\underline{d}| = 0$. Assume that one of the following conditions hold.*

(1) $d_Z \leq \delta_Z - 1$ for every proper subcurve $Z \subsetneq X$.
(2) X is semistable and $\underline{d}$ is balanced.
(3) $\delta_i \leq d_i \leq 2g_i - 2 + \delta_i$, for every $i = 1, \dots, \gamma$.

Then $h^0(X, L) \leq 1$ for every $L \in \operatorname{Pic}^{\underline{d}} X$.

Moreover, let $L \in \operatorname{Pic}^{\underline{d}} X$ be such that $h^0(L) = 1$. If (1) or (2) holds, or if (3) holds with $X_{sep} = \emptyset$, then $L \cong \mathcal{O}_X$.

Proof. This follows from Theorem 3.3.1, applying Riemann-Roch and Serre duality, together with some trivial arithmetic. ∎

4.2. **Clifford's theorem in degree at most** 4**.** The main result of this section is Theorem 4.2.8, stating the Clifford inequality for line bundles of balanced multidegree on semistable curves free from separating nodes. The proof is organized as follows. In Lemma 4.2.3, Lemma 4.2.4 and Proposition 4.2.5 we treat the case $\underline{d} \geq 0$, without assuming that $\underline{d}$ is balanced. The proof of Theorem 4.2.8 is thus reduced to assume that $\underline{d}$ has some negative entry.

Quite interestingly, if $d \geq 5$ Clifford's theorem fails even when X has no separating nodes. See Example 4.3.6.

4.2.1. Let $n \in X_{\text{sep}}$ be a separating node of X; then there exist two subcurves Z_1 and Z_2 of X such that $X = Z_1 \cup Z_2$ and $n = Z_1 \cap Z_2$. Such curves Z_1, Z_2 are called the tails of X generated by n. So, a subcurve $Z \subset X$ is called a *tail* if $Z \cdot Z^c = 1$. As X is connected, its tails are connected.

Let $C \subset X$ be a subcurve. C is called a *separating line* if $C \cong \mathbb{P}^1$ and if C meets its complementary curve C^c only in separating nodes of X. Equivalently: a separating line $C \subset X$ is a smooth rational component such that C^c has a number of connected components equal to $C \cdot C^c$.

If $X \cong \mathbb{P}^1$, then X is a separating line of itself.

If Y is a disconnected curve and $C \subset Y$, we say C is a separating line of Y if it is so for the connected component of Y containing C.

Observe that if C is a separating line, we have

$$Z \cdot C \leq 1 \quad \text{for every connected} \quad Z \subset C^c. \tag{38}$$

Remark 4.2.2. Assume $X_{\text{sep}} = \emptyset$; equivalently, assume that X has no tails. Let Z be a subcurve of X.

(A) If m is the number of connected components of Z, then $m \leq \frac{\delta_Z}{2}$

(B) Let $X = D \cup Y$ with D connected. If $C \subset Y$ is a separating line of Y, then $\overline{X \smallsetminus C}$ is connected.

The only statement that is not obvious is (B). Let $Y_1, \ldots, Y_m$ be the connected components of Y and suppose $C \subset Y_1$. We can assume $C \neq Y_1$. Thus every connected component of $\overline{Y_1 \smallsetminus C}$ is a tail of Y_1; as X has no tails D intersects every connected component of $\overline{Y_1 \smallsetminus C}$. On the other hand, D obviously intersects Y_i for all $i \geq 2$, therefore $\overline{X \smallsetminus C}$ is connected.

Lemma 4.2.3. *Let $L \in \operatorname{Pic}^{\underline{d}} X$. Assume $\underline{d} = (1, 0, \ldots, 0)$. Then either $h^0(X, L) \leq 1$, or C_1 is a separating line, $h^0(X, L) = 2$ and $L_{C_1^c} = \mathcal{O}_{C_1^c}$.*

Proof. Denote $Y = C_1^c$ and let $Y = \coprod_{i=1}^{c} Y_i$ be the decomposition into connected components. Of course C_1 must intersect every Y_i.

If $g_1 \geq 1$ we have $h^0(C_1, L_{C_1}) \leq 1$ hence the lemma follows from Remark 1.1.8 (with $V = C_1$). So it suffices to assume $C_1 \cong \mathbb{P}^1$. If C_1 is not a separating line there exists at least one connected component of Y, Y_1 say, such that $C_1 \cdot Y_1 \geq 2$. Set $X_1 = C_1 \cup Y_1$, then by Remark 1.1.8 and Lemma 1.1.9 we conclude as follows

$$h^0(X, L) \leq h^0(X_1, L_{X_1}) \leq h^0(C_1, L_1) + h^0(Y_1, L_{Y_1}) - 2 \leq 2 + 1 - 2 = 1.$$

If C_1 is a separating line and for some component of Y, Y_1 say, we have $L_{Y_1} \neq \mathcal{O}_{Y_1}$, then every section of L has to vanish on Y_1, hence not every section of $\mathcal{O}_{C_1}(1)$ extends to a section of L.

Conversely, if $L_{Y_i} = \mathcal{O}_{Y_i}$ for all i it is obvious that $h^0(X, L) = 2$. ■

Lemma 4.2.4. *Let $L \in \operatorname{Pic}^{\underline{d}} X$. Assume that $|\underline{d}| = 2$ and $\underline{d} \geq 0$. Then either $h^0(X, L) \leq 2$, or $h^0(X, L) = 3$ and one of the following cases occurs*

(i) $\underline{d} = (2, 0, \ldots, 0)$ with C_1 a separating line.

(ii) $\underline{d} = (1, 1, 0, \ldots, 0)$, with C_1 and C_2 separating lines.

Proof. Assume $h^0(L) \geq 3$. For every nonsingular point p of X we have

$$h^0(L(-p)) \geq h^0(L) - 1 \geq 2. \tag{39}$$

Of course, $\deg L(-p) = 1$ and, if p lies in a component C_1 such that $d_1 > 0$ we have $\underline{\deg} L(-p) \geq 0$. By Lemma 4.2.3 we get $h^0(L(-p)) \leq 1$, unless X has a separating line E with $\deg_E L(-p) = 1$. If X does not have such a separating line we got a contradiction to (39). Now, X admits such a separating line E if and only if either $d_1 = 2$ and $E = C_1$, or $d_1 = 1$, hence $d_2 = 1$, and C_2 is a separating line. By placing $p \in C_2$ we get that both C_1 and C_2 are separating lines. By Lemma 4.2.3 $h^0(L(-p)) = 2$, so $h^0(L) = 3$ by (39) and we are done. ■

Proposition 4.2.5. *Let X be a stable curve free from separating nodes. Let $\underline{d}$ be such that $\underline{d} \geq 0$ and $|\underline{d}| = 3, 4$. Then $h^0(X, L) \leq |\underline{d}|/2 + 1$ for every $L \in \operatorname{Pic}^{\underline{d}} X$.*

Remark 4.2.6. The hypotheses X stable and $X_{\text{sep}} = \emptyset$ are necessary, as shown by Examples 4.3.4 and 4.3.5.

Proof. We first treat the case $|\underline{d}| = 3$. Consider the irreducible component C_1 of X; we shall denote $C_1^c = Y_1 \coprod \ldots \coprod Y_m$ the connected component decomposition. Observe that for every Y_i we have $Y_i \cdot C_1 \geq 2$. We set

$$X_1 := C_1 \cup Y_1 \subset X.$$

We shall repeatedly apply Lemma 1.1.9 and Remark 1.1.8.

Case 1: $\underline{d} = (3, 0, \ldots, 0)$. We have $h^0(X, L) \leq h^0(X_1, L_{X_1})$ by Remark 1.1.8. Hence it suffices to assume that C_1 has genus $g_1 \leq 1$.

If $g_1 = 1$, by the initial observation and Lemma 1.1.9 we have $h^0(X_1, L_{X_1}) \leq 3 + 1 - 2 = 2$ and we are done.

If $C_1 \cong \mathbb{P}^1$ we have $h^0(C_1, L_1) = 4$ and $C_1 \cdot C_1^c \geq 3$. If C_1^c has a connected component, Y_1, such that $C_1 \cdot Y_1 \geq 3$, then $h^0(Y_1, L_{Y_1}) \leq 1$. By Lemma 1.1.9 we get $h^0(X_1, L_{X_1}) \leq 4 + 1 - 3 = 2$, as wanted.

Let now $C_1 \cdot Y_i = 2$ for all $i = 1, \ldots, m$. Set $X_2 = Y_1 \cup Y_2 \cup C_1 \subset X$. Then $C_1 \cdot (Y_1 \cup Y_2) \geq 4 = d_1 + 1$, hence by Lemma 1.1.9,

$$h^0(X_2, L_{X_2}) \leq h^0(C_1, L_1) + h^0(Y_1, L_{Y_1}) + h^0(Y_2, L_{Y_2}) - 4 \leq 4 + 2 - 4 = 2.$$

By Remark 1.1.8 we are done.

Case 2: $\underline{d} = (1, 2, 0, \ldots, 0)$.

Denote $l_i = h^0(C_i, L_i)$. Assume C_1^c connected; by Lemma 4.2.4, $h^0(C_1^c, L_{C_1^c}) \leq 3$ and equality holds if and only if C_2 is a separating line of C_1^c. If this is not the case, by Lemma 1.1.9 and $\delta_1 \geq 2$, we get $h^0(X, L) \leq l_1 + 2 - 2 \leq 4 - 2 = 2$, as wanted.

If C_2 is a separating line of C_1^c, then $l_2 = 3$, and C_2^c is connected, by Remark 4.2.2 (B); hence $h^0(C_2^c, L_{C_2^c}) \leq 2$. Since $\delta_2 \geq 3$ (as $d_2 = 2$) we obtain

$$h^0(X, L) \leq l_2 + h^0(C_2^c, L_{C_2^c}) - 3 \leq 5 - 3 = 2$$

and we are done. This part works regardless of C_1^c being connected.

Now let C_1^c have $m \geq 2$ connected components. We can assume that C_2 is not a separating line of C_1^c. Let $C_2 \subset Y_1$; we have $h^0(Y_1, L_{Y_1}) \leq 2$. By Lemma 1.1.9 we get $h^0(X_1, L_{X_1}) \leq h^0(C_1, L_1) + h^0(Y_1, L_{Y_1}) - 2 \leq 2$. By Remark 1.1.8 we are done.

Case 3: $\underline{d} = (1, 1, 1, 0, \ldots, 0)$. By Proposition 3.1.1 we may assume that $C_1 \cong \mathbb{P}^1$. Moreover, by Lemma 4.2.7, up to permuting the first three components, we can assume that C_2 and C_3 are not separating lines of C_1^c. If C_1^c is connected, by Lemma 4.2.4 we have $h^0(C_1^c, L_{C_1^c}) \leq 2$ (as C_2, C_3 are not separating lines of C_1^c). By Lemma 1.1.9 we have $h^0(X, L) \leq h^0(C_1, L_1) + h^0(C_1^c, L_{C_1^c}) - 2 \leq 2 + 2 - 2 \leq 2$ and we are done.

Now assume C_1^c has $m \geq 2$ connected components. If $C_2 \cup C_3$ lies in one connected component, Y_1, then $h^0(Y_1, L_{Y_1}) \leq 2$ (just as above). Therefore $h^0(X, L) \leq h^0(X_1, L_{X_1}) \leq 2 + 2 - 2 = 2$ ($X_1 = C_1 \cup Y_1$). If instead C_2 lies in Y_1 and C_3 lies in Y_2, then for $i = 1, 2$ we have $h^0(Y_i, L_{Y_i}) \leq 1$ by Lemma 4.2.3 (as C_2, C_3 are not separating lines of, respectively, Y_1, Y_2). We conclude $h^0(X_1, L_{X_1}) \leq 2 + 1 - 2 = 1$. Now, let $X_2 = X_1 \cup Y_2$, then

$$h^0(X, L) \leq h^0(X_2, L_{X_2}) \leq h^0(X_1, L_{X_1}) + h^0(Y_2, L_{Y_2}) \leq 2.$$

The proof for $d = 3$ is complete.

Now let $|\underline{d}| = 4$. By contradiction, suppose that $h^0(X, L) \geq 4$. As $\underline{d} \geq 0$, there exists a component, C_1 say, such that $d_1 \geq 1$. Let $p \in C_1$ be a nonsingular point of X, then $h^0(L(-p)) \geq h^0(L) - 1 \geq 3$. Now, $\deg L(-p) = 3$ and $\underline{\deg}\, L(-p) \geq 0$. By the previous part, $h^0(L(-p)) \leq 2$; impossible. ∎

In the proof we used the following combinatorial Lemma.

Lemma 4.2.7. *Let X be stable, $X_{sep} = \emptyset$,and C_1, C_2 two irreducible components of X. Assume C_2 is a separating line of C_1^c, and C_1 is a separating line of C_2^c (i.e. (C_1, C_2) is a $\mathcal{B}$-pair, see definition 5.2.1). Then for every other component D of X, C_1 and C_2 are not separating lines of D^c.*

Proof. Note that by Remark 4.2.2 (B), C_1^c and C_2^c are connected. Call $T_1, \ldots, T_t$ the tails of C_1^c generated by C_2. Thus $C_1^c = C_2 \cup T_1 \cup \ldots \cup T_t$, with $T_i \cap T_j = \emptyset$ and $T_i \cdot C_2 = 1$. As C_2^c is connected, C_1 must intersect every T_i. As C_1 is a separating line of C_2^c, we have

$$C_1 \cdot T_i = 1, \quad \forall i. \tag{40}$$

Let D be another component of X, assume $D \subset T_1$. Set $Z = C_2 \cup T_2 \cup \ldots \cup T_t$, so that $C_1^c = Z \cup T_1$, hence $\delta_{C_1} = Z \cdot C_1 + T_1 \cdot C_1 = Z \cdot C_1 + 1 \geq 3$, by (40) and the stability of X. We conclude $Z \cdot C_1 \geq 2$. This implies that C_1 cannot be a separating line of D^c, as Z is connected and $Z \subset D^c$ (cf. 4.2.1 (38)). The same argument with C_1 and C_2 switching roles yields that C_2 is not a separating line of D^c. ∎

Theorem 4.2.8. *Let X be a stable curve free from separating nodes. Let $\underline{d}$ be balanced with $0 < |\underline{d}| \leq 4$; let $L \in \operatorname{Pic}^{\underline{d}} X$. Then*

(i) $h^0(X, L) \leq |\underline{d}|/2 + 1$.

(ii) *If $|\underline{d}| = 1, 2$ and $h^0(X, L) = |\underline{d}|$, then $\underline{d} \geq 0$.*

If $|\underline{d}| = 1, 2$ the hypotheses on X can be weakened as follows.

Addendum 4.2.9. *If $|\underline{d}| = 1$ the same holds if X is semistable and has no separating lines. If $|\underline{d}| = 2$ the same holds if X is semistable and $X_{sep} = \emptyset$.*

Proof. If $\underline{d} \geq 0$ the statement follows from Lemmas 4.2.3, 4.2.4 and Proposition 4.2.5. So, assume $\underline{d} \not\geq 0$; set $d = |\underline{d}|$. We shall inductively define a useful subcurve $V \subseteq X$. Let $V_0 := Z_{\underline{d}}^-$ (see (36)). Now define $V_1 \subset X$

$$V_1 := V_0 \cup \bigcup_{C_i \cdot V_0 > d_i = 0} C_i;$$

so V_1 is the union of V_0 with all components of degree 0 which intersect V_0. Next

$$V_2 := V_1 \cup \bigcup_{\substack{C_i \not\subset V_1, d_i \leq 1, \\ C_i \cdot V_1 > d_i}} C_i.$$

Iterating

$$V_{h+1} := V_h \cup \bigcup_{\substack{C_i \not\subset V_h, d_i \leq h, \\ C_i \cdot V_h > d_i}} C_i \subset X.$$

Of course, $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_h \subseteq V_{h+1} \subseteq \ldots \subseteq X$, therefore there exists an $m \geq 0$ minimum for which $V_n = V_m$ for every $n \geq m$. We set $V := V_m$.

W e claim that every $s \in H^0(X, L)$ vanishes identically on V. It is clear that s vanishes on V_0; let us prove the claim inductively. Let $h \geq 0$ be such that V_{h+1} is not equal to V_h; by induction s vanishes identically on V_h. Let $C \subset V_{h+1}$ be such that C is not contained in V_h. Then s vanishes on $C \cap V_h$. Now, V_{h+1} is constructed so that $C \cdot V_h > \deg_C L > 0$, therefore s vanishes on C. The claim is proved.

If $V = X$ we have $H^0(X, L) = 0$ and we are done. So assume that $Y := V^c$ of V is not empty. Denote $G_Y \in \operatorname{Div} Y$ the divisor cut out by V, so that

$$\deg G_Y = \delta_Y. \tag{41}$$

Notice that

$$H^0(X, L) \cong H^0(Y, L_Y(-G_Y)). \tag{42}$$

By construction we have

$$\underline{d}_Y - \underline{\deg}\, G_Y \geq 0. \tag{43}$$

We claim that

$$0 \leq d_Y - \delta_Y \leq d - 2. \tag{44}$$

Set $a = d_Y - \delta_Y$. That $0 \leq a$ follows from (41) and (43). Now, notice that $w_Y < w$. Indeed, as $\underline{d}_V \not\geq 0$ by construction, $V = Y^c$ is not a union of exceptional components (see the initial observation). Hence (cf. 2.1) $w_V > 0$ and $w_Y = w - w_V < w$. As $\underline{d}$ is balanced, we obtain

$$\delta_Y + a = d_Y \leq \frac{\delta_Y}{2} + \frac{d w_Y}{w} < \frac{\delta_Y}{2} + d. \tag{45}$$

Therefore $\delta_Y \leq 2d - 2a - 1$. As $X_{\text{sep}} = \emptyset$ we have $\delta_Y \geq 2$. We obtain

$$2d - 2a - 1 \geq 2$$

hence $a \leq d - 3/2$, so that $a \leq d - 2$. (44) is proved.

We continue the proof with a case by case analysis.

Case $d = 1$. The inequality (44) makes no sense, hence Y is empty, i.e. $h^0(L) = 0$. We conclude that if $h^0(L) \neq 0$, then $\underline{d} \geq 0$, a case treated in Lemma 4.2.3. The assumptions X stable and $X_{\text{sep}} = \emptyset$ can clearly be weakened by, respectively, X semistable, and containing no separating line (needed for Lemma 4.2.3). If $d = 1$ the Theorem and the Addendum are proved.

Case $d = 2$. By (44) we have $d_Y = \delta_Y$, hence $\deg L_Y(-G_Y) = 0$. Now, using (45) we get $\delta_Y = d_Y < \frac{\delta_Y}{2} + 2$, hence $\delta_Y \leq 3$. This yields that Y is connected, by Remark 4.2.2 (A). We can apply Fact 1.1.7 to $L_Y(-G_Y)$, obtaining, with (42),

$$h^0(X, L) = h^0(Y, L_Y(-G_Y) \leq 1.$$

This concludes the proof if $d = 2$. We also showed that if $h^0(X, L) = 2$ then $\underline{d} \geq 0$. Observe that the argument works if X is semistable, so the Theorem and the Addendum are proved. The remaining cases will be treated similarly.

Case $d = 3$. By (44) we have two possibilities: either $\delta_Y = d_Y$ or $\delta_Y + 1 = d_Y$. If $\delta_Y = d_Y$ we have, using (45), $\delta_Y = d_Y < \frac{\delta_Y}{2} + 3$, hence $\delta_Y \leq 5$. Therefore Y has at most two connected components (by Remark 4.2.2 (A)). Let Y_i be a connected component of Y, then, by (43), $d_{Y_i} = \delta_{Y_i}$, and we can apply Fact 1.1.7 to $L_{Y_i}(-G_{Y_i})$ (with self-explanatory notation). Hence $h^0(Y_i, L_{Y_i}(-G_{Y_i}) \leq 1$; now Y has at most two connected components, hence by (42) we obtain $h^0(X, L) \leq 2$.

If $d_Y = \delta_Y + 1$, by (45) $\delta_Y + 1 = d_Y < \frac{\delta_Y}{2} + 3$, hence $\delta_Y \leq 3$, so Y is connected. By (43) and (44) we can apply Lemma 4.2.3 to $L_Y(-G_Y)$; we get

$$h^0(X, L) = h^0(Y, L_Y(-G_Y) \leq 2.$$

This finishes the proof in case $d = 3$.

Case $d = 4$. By (44) we have three possibilities: $d_Y = \delta_Y$, $d_Y = \delta_Y + 1$ or $d_Y = \delta_Y + 2$.

If $d_Y = \delta_Y$, we get $\delta_Y = d_Y < \frac{\delta_Y}{2} + 4$, hence $\delta_Y \leq 7$. Therefore Y has at most three connected components (again by Remark 4.2.2 (A)). Arguing as in the analogous case when $d = 3$ ($d_Y = \delta_Y$) we see that $h^0(X, L) \leq 3$ so we are done.

If $d_Y = \delta_Y + 1$, by (45) $\delta_Y + 1 = d_Y < \frac{\delta_Y}{2} + 4$, hence $\delta_Y \leq 5$ and Y has at most two connected components. If Y is connected arguing as in the analogous case when $d = 3$ we conclude $h^0(X, L) \leq 2$ and we are done. If Y has two connected components, Y_1 and Y_2, then we have $d_{Y_1} = \delta_{Y_1}$ and $d_{Y_2} = \delta_{Y_2} + 1$. We can therefore apply Fact 1.1.7 to get $h^0(Y_1, L_{Y_1}(-G_{Y_1})) \leq 1$, and 4.2.3 to get $h^0(Y_2, L_{Y_2}(-G_{Y_2})) \leq 2$. Summing up we obtain

$$h^0(X, L) = h^0(Y_1, L_{Y_1}(-G_{Y_1})) + h^0(Y_2, L_{Y_2}(-G_{Y_2})) \leq 3$$

and we are done. Finally, if $d_Y = \delta_Y + 2$, by the usual argument we get $\delta_Y \leq 3$ hence Y is connected. By Lemma 4.2.4 we have $3 \geq h^0(Y, L_Y(-G_Y)) = h^0(X, L)$ and we are done. ■

4.3. **Counterexamples.**

Example 4.3.1. *Failure of Clifford's theorem: $d = 1$, $\underline{d} \geq 0$ balanced (X contains a separating line).* Let $X = C_1 \cup C_2 \cup C_3 \cup C_4$ with, for $i, j \geq 2$, $C_i \cap C_j = \emptyset$ and $C_1 \cdot C_i = 1$ (the dual graph of X is in Figure 1). Assume $C_1 = \mathbb{P}^1$ (hence C_1 is a separating line) and $g_i = h \geq 1$ (hence X is stable). Thus $g = 3h$ and $w = 6h - 2$. Set $\underline{d} = (1, 0, 0, 0)$, one checks that $\underline{d} \in B_1(X)$. Let

$$L := (\mathcal{O}_{C_1}(1), \mathcal{O}_{C_2}, \mathcal{O}_{C_2}, \mathcal{O}_{C_4}).$$

Then, as all L_i are free from base points, we get $h^0(X, L) = \sum_1^4 h^0(C_i, L_i) - 3 = 2$.

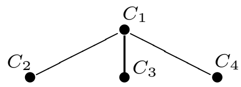

FIGURE 1. Dual graph of the curve in Example 4.3.1.

Example 4.3.2. Cliff $L = 0$ *with* $\underline{\deg} L \in B_1(X)$, $\underline{\deg} L \not\geq 0$ *($X_{sep} \neq \emptyset$).* Let $X = C_1 \cup C_2 \cup C_3$ with, $C_1 \cdot C_2 = 2$, $C_2 \cdot C_3 = 1$ and $C_1 \cap C_3 = \emptyset$ (see the picture below). Thus $n = C_2 \cap C_3$ is a separating node; for $i = 2, 3$, write $q_i \in C_i$ the point corresponding to this node. Assume $g_1 = g_2 = 1$ and $g_3 = 4$, thus $g = 7$. Set $\underline{d} = (1, -1, 1)$; one checks that $\underline{d} \in B_1(X)$. Call $Z = C_1 \cup C_2 \subset X$ and let $L_{1,2} \in \operatorname{Pic}^{(1,-1)} Z$ be arbitrary. Note that $h^0(Z, L_{1,2}) = 0$. Set

$$L := (L_{1,2}, \mathcal{O}_{C_3}(q_3)).$$

Then, as $L_{1,2}$ and $\mathcal{O}_{C_3}(q_3)$ both have a base point in the respective branch (q_2 and q_3) of n, we get $h^0(X, L) = h^0(Z, L_{1,2}) + h^0(C_3, \mathcal{O}_{C_3}(q_3)) = 1$.

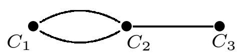

FIGURE 2. Dual graph of the curve in Example 4.3.2.

Example 4.3.3. *Failure of Clifford's theorem: $d = 2$, $\underline{d}$ balanced ($X_{sep} \neq \emptyset$).* Let $X = C_1 \cup C_2 \cup C_3 \cup C_4$ with, for $i, j \geq 2$, $C_i \cap C_j = \emptyset$ and $C_1 \cdot C_i = 1$ (same dual graph as in Figure 1). Let $g_1 = 1$ and $g_2 = g_3 = g_4 = 3$ so that $g = 10$. Let $\underline{d} = (-1, 1, 1, 1)$; one checks that $\underline{d}$ is the unique balanced multidegree of degree 2. Let L_1 be any line bundle of degree -1 on C_1. For $i = 2, 3, 4$ denote by $q_i \in C_i$ the point corresponding to the node $C_1 \cap C_i$. Consider the degree 2 line bundle on X

$$L = (L_1, \mathcal{O}_{C_2}(q_2), \mathcal{O}_{C_3}(q_3), \mathcal{O}_{C_4}(q_4)).$$

As every section of $\mathcal{O}_{C_i}(q_i)$ vanishes in q_i, we get that $H^0(X, L) = 3$.

Example 4.3.4. *Failure of Clifford's theorem: $d \geq 3$, $\underline{d}$ balanced, $X_{sep} = \emptyset$ (X strictly semistable).* For $d \geq 3$ consider the curve $X = C_1 \cup \ldots \cup C_{2d}$ whose dual graph is a $2d$-cycle, i.e. a closed polygon with $2d$ vertices, $C_1, \ldots, C_{2d}$. We set $C_i \cdot C_{i+1} = C_{2d} \cdot C_1 = 1$ for all $i \geq 1$ and $C_i \cdot C_j = 0$ for all other intersections. So X has $2d$ nodes. Let $C_{2i-1} \cong \mathbb{P}^1$ for all i, so that the odd indexed components are exceptional; now let all the even indexed components be smooth of genus 1. Therefore $g = d + 1$. Now choose the multidegree $\underline{d} = (1, 0, 1, \ldots, 1, 0)$ and set $L_{C_{2h}} \cong \mathcal{O}_{C_{2h}}$ for all h (of course $L_{C_{2h+1}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$). One easily checks that $\underline{d}$ is balanced. It is also clear that for any $L \in \operatorname{Pic} X$ whose restrictions to the C_i are as above, we have $h^0(X, L) \geq 2d + d - 2d = d$. So Clifford's inequality fails.

Example 4.3.5. *Failure of Clifford's theorem: $d \geq 3$, $\underline{d} \geq 0$, $X_{sep} \neq \emptyset$.* Let $X = C_1 \cup C_2 \cup C_3$ with C_1 of genus 1 and $g_i \geq 1$. Let $C_1 \cdot C_2 = C_1 \cdot C_3 = 1$ and $C_2 \cdot C_3 = 0$ (the dual graph of X is obtained from the graph in Figure 1 by removing the vertex C_4 and the edge adjacent to it). Let $L = (L_1, \mathcal{O}_{C_2}, \mathcal{O}_{C_3}) \in \operatorname{Pic}^d X$ with $\deg L_1 = d$. Then $h^0(L) = d$.

Example 4.3.6. *Failure of Clifford's theorem:* $d = 5$, $\underline{d}$ *balanced and* $X_{sep} = \emptyset$. Let $X = C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$ with, for $i, j \geq 2$, $C_i \cap C_j = \emptyset$ and $C_1 \cdot C_i = 2$ for all $i \geq 2$. So every node of X lies on C_1, and $\delta = 8$ (the dual graph of X is in Figure 3). Now let h be any nonnegative integer. Let C_1 be of genus $g_1 = h$, and let C_i have genus $h+3$ for every $i \geq 2$. Hence $g = 5h + 16$. We now pick $d = 5$ and $\underline{d} = (-3, 2, 2, 2, 2)$. It is straightforward to check that $\underline{d}$ is balanced.

Now for $i \geq 2$, set $\{p_i, q_i\} = C_1 \cap C_i \subset C_i$. Let L be any line bundle whose restrictions $(L_1, \ldots, L_5)$ are as follows. $L_1 \in \operatorname{Pic}^{-3} C_1$ is arbitrary, while $L_i = \mathcal{O}_{C_i}(p_i + q_i)$, for $i = 2, 3, 4, 5$.

Now every section s of L vanishes identically on C_1, hence s vanishes on p_i, q_i. Conversely, any quadruple of sections $s_i \in H^0(C_i, L_i(-p_i - q_i))$, for $i = 2, \ldots, 5$, glues to a section of L. We conclude $h^0(X, L) = \sum_{i=2}^{5} h^0(C_i, L_i(-p_i - q_i)) = 4$. So L violates Clifford inequality. Similar examples exist for higher degree d.

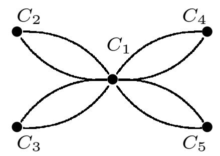

FIGURE 3. Dual graph of the curve in Example 4.3.6.

5. APPLICATIONS

If $g \geq 3$ we denote by $\overline{H}_g \subset \overline{M}_g$ the closure of the locus of hyperelliptic curves. Recall that $\overline{H}_g$ is an irreducible subscheme of dimension $2g-1$. Following a common practice (see [HM82]), we say that a stable curve X is *hyperelliptic* if $[X] \in \overline{H}_g$.

Definition 5.0.7. We call a stable curve X *weakly hyperelliptic* if there exists a balanced line bundle $L \in \operatorname{Pic}^2 X$ such that $h^0(X, L) \geq 2$.

Lemma 5.0.8. *If X is hyperelliptic, X is weakly hyperelliptic.*

Remark 5.0.9. The converse is false, see Remark 5.1.4.

Proof. As $[X] \in \overline{H}_g$ there exists a one parameter smoothing of X, $f : \mathcal{X} \to \operatorname{Spec} R$, whose generic fiber is a smooth hyperelliptic curve. We can also assume that $\mathcal{X}$ is regular, and that there exists $\mathcal{L} \in \operatorname{Pic} \mathcal{X}$ such that the restriction of $\mathcal{L}$ to the generic fiber is the hyperelliptic bundle. Set $L = \mathcal{L}_{|X}$. Up to tensoring $\mathcal{L}$ with a divisor supported entirely on the closed fiber X we can assume that L is balanced. By uppersemicontinuity of h^0 we have $h^0(X, L) \geq 2$, so we are done. ∎

5.1. **Clifford index of two-components curves.** Recall that smooth hyperelliptic curves can be characterized using Clifford's inequality; the same holds for irreducible curves (see [C07, Sec. 5]). We shall generalize this to stable curves having two components. So, let $X = C_1 \cup C_2$ have genus $g \geq 2$; we proved in Theorem 3.2.1 that the Clifford's inequality holds.

The Clifford index of a line bundle has been introduced in 1.2. Now, if X is irreducible, its Clifford index is defined as $\operatorname{Cliff} X = \min\{\operatorname{Cliff} L\}$ where L varies in the set of line bundles on X such that $h^0(X, L) \geq 2$ and $h^1(X, L) \geq 2$. By Clifford's theorem, $\operatorname{Cliff} X \geq 0$; moreover, $\operatorname{Cliff} X = 0$ if and only if X is hyperelliptic. We extend the definition of the Clifford index to a semistable curve X as follows.

$$\operatorname{Cliff} X = \min\{\operatorname{Cliff} L \mid \underline{\deg} L \in B_d(X),\ h^0(X, L) \geq 2,\ h^1(X, L) \geq 2\}. \tag{46}$$

By Theorem 3.2.1, $\operatorname{Cliff} X \geq 0$ if $X = C_1 \cup C_2$. We now ask: when is $\operatorname{Cliff} X = 0$? To answer this question we use the following terminology. A curve X (reduced, nodal, of genus g) is called a *binary curve* if it is the union of two copies of $\mathbb{P}^1$ meeting transversally in $g+1$ points (cf. [C08]).

Proposition 5.1.1. *Let $X = C_1 \cup C_2$ be semistable.*

(1) $\operatorname{Cliff} X = 0$ if and only if X is weakly hyperelliptic.

(2) If X is weakly hyperelliptic, then $C_1 \cdot C_2 \leq 2$ unless X is a hyperelliptic binary curve.

Proof. As we said, Theorem 3.2.1 yields $\operatorname{Cliff} X \geq 0$. Therefore if X is weakly hyperelliptic, then $\operatorname{Cliff} X = 0$.

Conversely, suppose $\operatorname{Cliff} X = 0$; let $L \in \operatorname{Pic}^{\underline{d}}(X)$ with $\underline{d} \in B_d(X)$, such that $h^0(L) = d/2 + 1$. If $d = 2$ there is nothing to prove, so assume $d > 2$. As usual, set $\delta = C_1 \cdot C_2$. We must prove that there exists a $J \in \operatorname{Pic}^2 X$ such that $h^0(J) = 2$ and $\underline{\deg}\, J \in B_2(X)$.

• Assume first $d_i \leq 2g_i$ for $i = 1, 2$. By Corollary 3.1.2 we have $\delta \leq 2$.

Suppose $\delta = 2$; again by Corollary 3.1.2 we have $\operatorname{Cliff} L_1 = \operatorname{Cliff} L_2 = 0$ and, if $d_i \geq 2$, then $\operatorname{Cliff} L_i(-C_1 \cap C_2) = 0$.

If $d_1 = 0$ then $L_1 = \mathcal{O}_{C_1}$ and $L_2 = H_2^{d/2}$ for some $H_2 \in W_2^1(C_2)$ (see [C07, subsec. 5.2]). By hypothesis $(0, d) \in B_d(X)$, which easily implies that $g_2 > g_1$, and hence that multidegree $(0, 2)$ is balanced. Consider the line bundle $M := (\mathcal{O}_{C_1}, H_2)$ on the normalization X^ν of X; as $\operatorname{Cliff} H_2^{d/2}(-C_1 \cap C_2) = 0$ we have $h^0(C_2, H_2(-C_1 \cap C_2)) = 1$, hence by Lemma 1.1.5 there exists $J \in F_M(X)$ such that $h^0(X, J) = h^0(X^\nu, M) - 1 = 2$. Since $\underline{\deg}\, J = (0, 2)$ is balanced we are done.

If $d_i > 0$ for $i = 1, 2$ then there exists $H_i \in W_2^1(C_i)$ such that $L_i = H_i^{d_i/2}$, for both i. Suppose $g_1 \leq g_2$; arguing as above we see that $(0, 2)$ is balanced and that there exists $J \in W^1_{(0,2)}(X)$ such that the pull-back of J to the normalization of X is $(\mathcal{O}_{C_1}, H_2)$. Up to switching C_1 and C_2 we are done.

Suppose $\delta = 1$. If $(1, 1)$ is balanced, then X is (trivially) weakly hyperelliptic (see Lemma 5.1.3). So assume $(1, 1)$ not balanced. By Example 2.1.2 we may assume $g_1 < g_2$ and $B_2(X) = \{(0, 2)\}$. By Corollary 3.1.2, $\operatorname{Cliff} L_2 = 0$, therefore C_2 is hyperelliptic. Let H_{C_2} be its hyperelliptic bundle, and set $J = (\mathcal{O}_{C_1}, H_2)$; it is clear that $h^0(X, J) = 2$.

• Now assume that $d_1 = 2g_1 + e$ with $e \geq 1$. We will prove that X is a binary curve. In this case the result is known: a binary curve is hyperelliptic if and only if it is weakly hyperelliptic ([C08, Sec. 3]).

We are in the situation treated in the proof of 3.2.1, from which we now use the notation. We saw there that the Clifford inequality can be an equality only in Case 2, at the very end. More precisely, in order for $\operatorname{Cliff} L = 0$ we must have $d_2 = 2g_2 + e$ (so that $d = 2g_1 + 2g_2 + 2e$) and

$$l = l_1 + l_2 - e - 1. \tag{47}$$

Now, as $d < 2g - 2$ and $g = g_1 + g_2 + \delta - 1$ we have $2(g_1 + g_2 + e) < 2(g_1 + g_2 + \delta - 2)$, hence

$$e \leq \delta - 3. \tag{48}$$

Now let $\beta := e + 1$, so that $\beta \leq \delta - 2$. Set

$$Y = (C_1 \coprod C_2)/_{\{p_i = q_i,\ \ i=1,\dots,\beta\}} \xrightarrow{\nu} X,$$

i.e. ν is the normalization of X at $\delta - \beta$ nodes. Let $M = \nu^* L$; we have, by Lemma 1.1.9 (ii),

$$h^0(Y, M) = l_1 + l_2 - e - 1 = l = h^0(X, L)$$

using (47). Therefore for all $i = \beta+1, \dots, \delta$, we have $p_i \sim_M q_i$, by Lemma 1.1.5. This implies that, for all $i \geq \beta+1$, p_i is a base point of $L_1(-\sum_{j=1}^{\beta} p_j)$ and q_i is a base point of $L_2(-\sum_{j=1}^{\beta} q_j)$ (by Lemma 1.1.4). Now

$$\deg L_1(-\sum_{j=1}^{\beta} p_j) = 2g_1 + e - \beta = 2g_1 - 1, \quad \deg L_2(-\sum_{j=1}^{\beta} q_j) = 2g_2 + e - \beta = 2g_2 - 1.$$

If X is not a binary curve, we may assume $g_2 \geq 1$. Then, $L_2(-\sum_{j=1}^{\beta} q_j)$, having degree $2g_2 - 1$, can have at most one base point. Therefore $\delta - \beta \leq 1$, i.e. $\delta - e \leq 2$, which is in contradiction with (48). We conclude that X is a binary curve. ∎

5.1.2. *Curves of compact type* For any integer h with $1 \leq h \leq g/2$, let Δ_h be the divisor in $\overline{M}_g$ whose general point represents a curve $X = C_1 \cup C_2$ with C_i smooth, $C_1 \cdot C_2 = 1$ and $g_1 = h$. Fix such an X; for $i = 1, 2$ we shall denote by $q_i \in C_i$ the branches of the node of X. We computed $B_2(X)$ in Example 2.1.2.

Lemma 5.1.3. *Let $X = C_1 \cup C_2$ with $C_1 \cdot C_2 = 1$ and $1 \leq g_1 \leq g/2$.*

Let $g_1 \geq (g+1)/4$. Then X is weakly hyperelliptic; more precisely, $(1,1)$ *is balanced and $W^1_{(1,1)}(X) = \{(\mathcal{O}_{C_1}(q_1), \mathcal{O}_{C_2}(q_2)\}$.*

Let $g_1 < (g+1)/4$. Then X is weakly hyperelliptic if and only if C_2 is hyperelliptic, if and only if $W^1_{(0,2)}(X) = \{(\mathcal{O}_{C_1}, H_{C_2})\}$.

Proof. Set $L = (\mathcal{O}_{C_1}(q_1), \mathcal{O}_{C_2}(q_2)) \in \operatorname{Pic} X$. It is clear that $h^0(X, L) = 2$. If $g_1 \geq (g+1)/4$, then L is balanced. Conversely, let $L' \in W^1_{(1,1)}(X)$; by Corollary 3.1.2 we have $L' = (\mathcal{O}_{C_1}(q_1), \mathcal{O}_{C_2}(q_2))$, so the first part is proved.

Now suppose $g_1 < (g+1)/4$, then $(0,2)$ is the unique balanced multidegree. If C_2 is hyperelliptic, the balanced line bundle $L = (\mathcal{O}_{C_1}, H_{C_2}) \in \operatorname{Pic} X$ has, of course, $h^0(X, L) = 2$. So, X is weakly hyperelliptic. Conversely, if there exists $L \in \operatorname{Pic}^{(0,2)} X$ such that $h^0(L) = 2$, we can apply Corollary 3.1.2 (we necessarily have $g_2 \geq 3$ by hypothesis) and conclude that $h^0(C_2, L_2) = 2$, so we are done. ∎

Remark 5.1.4. The previous result shows that there exist (plenty of) weakly hyperelliptic curves that are not hyperelliptic. Indeed, it is well known that a curve of compact type $X = C_1 \cup C_2$ is hyperelliptic if and only if both C_1 and C_2 are hyperelliptic, and the two branches, q_1 and q_2, are Weierstrass points (cf. [CH88] for example). Also, there exist globally generated balanced line bundles $L \in W^1_2(X)$ which are not limits of hyperelliptic bundles of smooth curves (indeed $(\mathcal{O}_{C_1}, H_{C_2})$ is always globally generated).

5.2. **Hyperelliptic and weakly hyperelliptic curves.** The next definition will be used only when $X_{\text{sep}} = \emptyset$.

Definition 5.2.1. A pair (C, D) of (smooth, rational) components of X is called a *binary-pair* (or a *$\mathcal{B}$-pair* for short) of X if C is a separating line of D^c and D is a separating line of C^c. The subcurve $C \cup D$ will be called a *$\mathcal{B}$-subcurve*.

Example 5.2.2. Let X be a binary curve (defined before Proposition 5.1.1); then its irreducible components form a $\mathcal{B}$-pair. Also, if $X' = C \cup D \cup E_1 \cup \ldots \cup E_s$ is a semistable curve whose stabilization is a binary curve $X = C \cup D$, then (C, D) is a $\mathcal{B}$-pair of X' .

Let (C, D) be a binary pair of X. Denote $C \cap D = \{n_1, \ldots, n_l\}$, with $l \geq 0$, and $q^i_C \in C$, $q^i_D \in D$ the two branches of n_i. If $C \cup D \neq X$, there is a decomposition $X = (C \cup D) \cup (Z_1 \coprod \ldots \coprod Z_m)$ where Z_j are connected and $Z_j \cdot C = Z_j \cdot D = 1$

for all j. Denote $p_C^j = C \cap Z_j$ and $p_D^j = D \cap Z_j$. Let $n = l + m$ ($m \geq 0$); now the ordered n-tuples

$$G_C := (q_C^1, \dots, q_C^l, p_C^1 \dots, p_C^m) \subset C, \quad G_D := (q_D^1, \dots, q_D^l, p_D^1 \dots, p_D^m) \subset D \tag{49}$$

give a structure of n-marked curve on C and D. We say that (C, D) is a *special $\mathcal{B}$-pair* if $(C; G_C)$ and $(D; G_D)$ are isomorphic as n-marked curves.

Theorem 5.2.3. *Let X be semistable with $X_{sep} = \emptyset$; let $\underline{d}$ be such that $|\underline{d}| = 2$. Assume that $\underline{d}$ is balanced, or that X stable and $\underline{d} \geq 0$. Suppose there exists $L \in \operatorname{Pic}^{\underline{d}} X$ with $h^0(X, L) = 2$.*

Then L is globally generated, and either one of the two cases below occurs.

(1) $\underline{d} = \underline{(1, 1, 0 \dots 0)}$ and (C_1, C_2) is a special $\mathcal{B}$-pair of X. Also, the restriction of L to $\overline{X \smallsetminus (C_1 \cup C_2)}$ is trivial.

(2) $\underline{d} = (2, 0, \dots, 0)$ and, denoting $C_1^c = Z_1 \coprod \dots \coprod Z_m$, with Z_i connected, $\forall i = 1 \dots m$ we have

$$C_1 \cdot Z_i = 2, \quad L_{C_1} \cong \mathcal{O}_{C_1}(C_1 \cap Z_i), \quad L_{C_1^c} \cong \mathcal{O}_{C_1^c} \quad \textit{and} \quad h^0(C_1, L_{C_1}) \geq 2.$$

Conversely, if X and $\underline{d}$ satisfy the above properties, there exists a unique line bundle $L \in \operatorname{Pic}^{\underline{d}} X$ such that $W^1_{\underline{d}}(X) = \{L\}$.

Proof. Assume there exists $L \in W^1_{\underline{d}}(X)$; by Theorem 4.2.8 (ii), $\underline{d} \geq 0$. By the same Theorem and its addendum we obtain that L is globally generated.

We fix C a non-disconnecting component of X (Remark 3.3.4), and set $Z = C^c$.

Step 1. *Case* $(d_C, d_Z) = (1, 1)$.

Let $D \subset Z$ be the component such that $d_D = 1$. We must prove that (C, D) is a $\mathcal{B}$-pair of X.

By contradiction, suppose D is not a separating line of Z; this implies $h^0(Z, L_Z) \leq 1$ (by Lemma 4.2.3). Let $C \not\cong \mathbb{P}^1$, then $h^0(C, L_C) \leq 1$. So, in order to have $h^0(X, L) = 2$ we must have $h^0(C, L_C) = h^0(Z, L_Z) = 1$ and every point in $Z \cap C \subset C$ must be a base point for L_C (by Lemma 1.1.5). This is impossible, as $Z \cdot C \geq 2$ and $d_C = 1$. Now let $C \cong \mathbb{P}^1$, hence $h^0(C, L_C) = 2$. By Lemma 1.1.9 we have

$$h^0(X, L) \leq h^0(C, L_C) + h^0(Z, L_Z) - 2 \leq 2 + 1 - 2 = 1;$$

a contradiction.

Therefore D is a separating line of Z, and $h^0(Z, L_Z) = 2$. By Remark 4.2.2 (B), D is a non-disconnecting component of X. So, we can switch C with D and, by the previous argument, we obtain that C is a separating line of D^c. In other words, (C, D) is a $\mathcal{B}$-pair of X, as stated.

Now, as $h^0(L) = 2$, the restriction of L to the complement of $C \cup D$ is trivial. Therefore L determines a map ψ to $\mathbb{P}^1$ such that $\psi(p_C^j) = \psi(p_D^j)$ for all j (notation as in (49)). Hence ψ induces an isomorphism of the n-marked curves C, D with the same n-marked $\mathbb{P}^1$. This shows that the pair (C, D) is special.

Step 2. *Case* $(d_C, d_Z) = (2, 0)$.

Now X must be a stable curve (an exceptional component must have degree 1). We must prove that $L_Z \cong \mathcal{O}_Z$, that $C \cdot Z = 2$ and that, setting $C \cap Z = \{p, q\} \subset C$, we have $\mathcal{O}_C(p + q) \cong L_C$. Assume first $C \not\cong \mathbb{P}^1$. So $h^0(C, L_C) \leq 2$ with equality only if L_C has no base point; also, $h^0(Z, L_Z) \leq 1$ with equality if and only if $L_Z = \mathcal{O}_Z$ (by Fact 1.1.7). It is clear that, for $h^0(X, L) = 2$, we must have equality in both cases. Hence $L_Z = \mathcal{O}_Z$. If $C \cdot Z \geq 3$, by Lemma 1.1.6 there exist three points $p, q, r \in C$ such that

$$p \sim_{L_C} q \sim_{L_C} r.$$

Now L_C has no base points, hence we get

$$1 = h^0(C, L_C) - 1 = h^0(C, L_C(-p)) = h^0(C, L_C(-p - q - r))$$

which is impossible, as $\deg L_C(-p-q-r)=-1$. We thus proved that $C\cdot Z=2$, that $h^0(C,L_C(-p-q))=1$, i.e. $L_C=\mathcal{O}_C(p+q)$. The uniqueness of L follows from Lemma 1.1.5.

Now let us prove that $C\ncong\mathbb{P}^1$. By contradiction, if $C\cong\mathbb{P}^1$, then $\delta_C\geq 3$ (X is stable) and $h^0(C,L_C)=3$. By Lemma 1.1.9 we obtain

$$h^0(X,L)\leq h^0(C,L_C)+h^0(Z,L_Z)-3\leq 3+1-3=1$$

which is impossible.
Step 3. *Case* $(d_C,d_Z)=(0,2)$.

Now $h^0(C,L_C)\leq 1$ with equality if and only if $L_C=\mathcal{O}_C$. Suppose Z contains a separating line E such that $d_E\geq 1$; then E^c is connected (by Remark 4.2.2(B)); we may thus replace C with E, and be back in the situations treated in the previous steps.

So, we are reduced to assume Z contains no such separating line. By Lemma 4.2.4, and because $h^0(X,L)=2$, we have $h^0(Z,L_Z)=2$ and $L_C=\mathcal{O}_C$. Let $D\subset Z$ be an irreducible component with $d_D\geq 1$. Denote $Y=D^c=Y_1\coprod\ldots\coprod Y_m$ the connected components decomposition. We have ($X_{\text{sep}}=\emptyset$)

$$D\cdot Y_i\geq 2,\quad\forall i=1,\ldots,m. \tag{50}$$

Assume $d_D=1$. Let Y_1 be the connected component such that $d_{Y_1}=1$; then $h^0(Y_1,L_{Y_1})\leq 1$ (by Lemma 4.2.3, as Y_1 contains no separating line having degree 1). Therefore, setting $X_1=D\cup Y_1\subset X$, Lemma 1.1.9 yields

$$h^0(X_1,L_{X_1})\leq 2+1-2=1.$$

As $h^0(X,L)\leq h^0(X_1,L_{X_1})$ (by Remark 1.1.8) we have a contradiction.

Therefore we must have $d_D=2$, hence $\underline{d}_{D^c}=\underline{0}$. Now, for every $i=1,\ldots,m$ we argue as in Step 2, with $X_i=D\cup Y_i$ playing the role of X, D playing the role of C, and Y_i playing the role of Z. This shows that L is unique and that for every i, D intersects Y_i in two points $p_i,q_i\in D$, that $L_D\cong\mathcal{O}_D(p_i+q_i)$ and that $L_{Y_i}\cong\mathcal{O}_{Y_i}$. This concludes Step 3.

The converse follows easily from Lemma 1.1.5. The proof is complete. ■

Remark 5.2.4. Let X be a stable curve such that $X_{\text{sep}}=\emptyset$. Then X admits a decomposition (unique up to the order) $X=A_1\cup\ldots\cup A_\alpha$ such that every A_i is either a $\mathcal{B}$-subcurve or an irreducible component of X not part of any $\mathcal{B}$-pair

This follows from the fact that, by Lemma 4.2.7, every irreducible component of X belongs to at most one $\mathcal{B}$-pair.

Proposition 5.2.5. *Let X be a hyperelliptic stable curve such that $X_{sep}=\emptyset$. Consider the decomposition $X=A_1\cup\ldots\cup A_\alpha$ defined in Remark 5.2.4. Then for every $i\neq j$ we have either $A_i\cap A_j=\emptyset$, or*

$$A_i\cdot A_j=2\quad\textit{and}\quad h^0(A_i,\mathcal{O}_{A_i}(A_i\cap A_j))\geq 2.$$

Proof. We begin as in the proof of Lemma 5.0.8 Let $f:\mathcal{X}\to B$ be a regular one-parameter smoothing of X, whose generic fiber is hyperelliptic, and let $\mathcal{L}\in\operatorname{Pic}\mathcal{X}$ be a balanced line bundle such that the restriction of $\mathcal{L}$ to the generic fiber is the hyperelliptic bundle, set $\mathcal{L}_{|X}=L$. Now for every divisor $T\in\operatorname{Div}\mathcal{X}$ supported on X, denote

$$L_T:=\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}(T)\otimes\mathcal{O}_X.$$

For every T we have $\deg L_T=2$ and, by uppersemicontinuity of h^0, $h^0(X,L_T)\geq 2$.

By assumption $\underline{d}=\underline{\deg}\,L$ is balanced. By Theorem 5.2.3 we have $X=A\cup(Z_1\coprod\ldots\coprod Z_m)$ where A is either an irreducible component, in which case $\underline{d}_A=2$, or a $\mathcal{B}$-subcurve, in which case $\underline{d}_A=(1,1)$. Recall that $Z_i\cap Z_j=\emptyset$, $Z_i\cdot A=2$, and

that if A is a $\mathcal{B}$-pair then $\underline{\deg}_A Z_i = (1,1)$. We also have (always by Theorem 5.2.3) $L_{A^c} = \mathcal{O}_{A^c}$ and $h^0(A, A \cap Z_i) \geq 2$ for every i. Set $A = A_1$.

Consider L_T with $T = -Z_1$. By what we just said $\underline{\deg}\, L_T \geq 0$, indeed for any component (or subcurve) $C \subset Z_1$ we have $\deg_C L_T = -\deg_C Z_1 = C \cdot A_1 \geq 0$; if instead $C \subset Z_1^c$ then $\deg_C L_T = 0$. We can apply thus Theorem 5.2.3 to L_T. Since $\underline{\deg}_{A_1} L_T = \underline{0}$ and $\underline{\deg}_{Z_i} L_T = \underline{0}$ if $i \neq 1$, we derive that Z_1 contains a subcurve A_2 with the same properties as A_1; in particular, A_2 is either irreducible or a $\mathcal{B}$-subcurve, and $A_1 \cdot A_2 = 2$, because $\deg_{A_2} L_T = 2$. Therefore $A_1 \cap Z_1 = A_1 \cap A_2$ and $h^0(A_1, A_1 \cap A_2) \geq 2$. Thus the part of the statement concerning A_1 and A_2 is satisfied; so, if $A_2 = Z_1$ we turn to Z_i with $i \geq 2$. If instead $A_2 \subsetneq Z_1$, we iterate the procedure with A_2 as the starting component and $T = -W$ with W a connected component of $\overline{Z_1 \smallsetminus A_2}$. Obviously this iteration stops after finitely many steps. By repeating this argument for every Z_i we are done. ■

5.3. Curves of genus 6 admitting a g^2_5.

5.3.1. Throughout this subsection we shall consider curves $X = C_1 \cup C_2$, of genus 6, such that C_1 and C_2 are smooth, of respective genus g_1 and g_2; we set $\delta = C_1 \cdot C_2$. For any $L \in \operatorname{Pic} X$ we write $L_i = L_{|C_i}$ and $h^0(L_i) = h^0(C_i, L_i)$. We fix points $p_1, \ldots, p_\delta \in C_1$ and $q_1, \ldots, q_\delta \in C_2$ so that $X = (C_1 \coprod C_2)/_{(p_i = q_i,\ \ i=1,\ldots,\delta)}$ and set

$$G_1 := \sum_{i=1}^{\delta} p_i, \quad G_2 := \sum_{i=1}^{\delta} q_i. \tag{51}$$

Finally, we set $\underline{g} := (g_1, g_2)$, and we always assume $g_1 \leq g_2$.

Theorem 5.3.2. *With the above set-up, let $X = C_1 \cup C_2$ be semistable of genus 6, and let $\underline{d} \in B_5(X)$. Assume there exists a globally generated $L \in W^2_{\underline{d}}(X)$. Then*

(I) If $\delta = 1$, C_2 is not hyperelliptic and one of the following cases occurs.
  - *(a) $\underline{g} = (1,5)$, $\underline{d} = (0,5)$, $L_1 = \mathcal{O}_{C_1}$, and $h^0(L_2) = 3$.*
  - *(b) $\underline{g} = (2,4)$ or $\underline{g} = (3,3)$, $\underline{d} = (2,3)$, and $h^0(L_1) = h^0(L_2) = 2$.*

(II) If $\delta = 2$ one of the following cases occurs.
  - *(a) $\underline{g} = (0,5)$, $\underline{d} = (1,4)$, C_2 hyperelliptic, $L_2 = H_{C_2}^{\otimes 2}$.*
  - *(b) $\underline{g} = (1,4)$, $\underline{d} = (0,5)$, $L_1 = \mathcal{O}_{C_1}$, C_2 not hyperelliptic, $h^0(L_2) = 3$.*
  - *(c) $\underline{g} = (2,3)$, $\underline{d} = (2,3)$, $L_1 = H_{C_1} = \mathcal{O}_{C_1}(G_1)$, C_2 not hyperelliptic, $L_2 = \mathcal{O}_{C_2}(G_2 + q)$ and $h^0(L_2) = 2$.*
  - *(d) $\underline{g} = (1,4)$ or $\underline{g} = (2,3)$, $\underline{d} = (2,3)$, $L_1 = \mathcal{O}_{C_1}(G_1)$, C_2 not hyperelliptic, $L_2 = \mathcal{O}_{C_2}(G_2 + q)$ and $h^0(L_1) = h^0(L_2) = 2$.*

(III) If $\delta = 3$ then $\underline{g} = (1,3)$ and one of the following cases occurs.
  - *(a) $\underline{d} = (3,2)$, $L_1 = \mathcal{O}_{C_1}(G_1)$, C_2 is hyperelliptic, $L_2 = H_{C_2}$.*
  - *(b) $\underline{d} = (0,5)$, $L_1 = \mathcal{O}_{C_1}$, and $h^0(L_2) = 3$.*

(IV) If $\delta = 4$, then $\underline{g} = (0,3)$, $\underline{d} = (1,4)$ and $L_2 = K_{C_2} = \mathcal{O}_{C_2}(G_2)$.

(V) If $\delta = 6$, then $\underline{g} = (0,1)$, $\underline{d} = (2,3)$.

Remark 5.3.3. The cases (I) and (II), i.e $\delta \leq 2$, are contained in Propositions 5.3.5 and 5.3.6, where a more precise statement is proved.

Proof. Our curve X has a priori $\delta \leq 7$ nodes. The case that $\delta = 7$, i.e. X is a binary curve, is ruled out as follows. Proposition 12 in [C08] implies $\underline{\deg}\, L = (2,3)$; by Proposition 19 and Lemma 20 in loc. cit. the curve X must be hyperelliptic. Therefore the canonical morphism maps X two-to-one onto a rational normal quintic in $\mathbb{P}^5$. Now we argue as for smooth curves (cf. [ACGH] D-9 p. 41): we have $h^0(X, \omega_X \otimes L^{-1}) = 3$, hence (as points on a rational normal curve are in general linear position) we easily get $L \cong H_X^{\otimes 2}(p)$ with $p \in X$ a base point of L. So L is not globally generated, and we are done.

From now on, by Remark 5.3.3, we assume $3 \leq \delta \leq 6$.

Pick $\underline{d}$ and $L \in W^2_{\underline{d}}(X)$ as in the statement. The fact that $\underline{d}$ is balanced means

$$g_i - 1 \leq d_i \leq g_i - 1 + \delta, \quad i = 1, 2, \tag{52}$$

and $d_i = 1$ if C_i is an exceptional component.

Let us, first of all, show that $\underline{d} \geq 0$. If $d_1 < 0$ we must have $\underline{d} = (-1, 6)$, and $g_1 = 0$. We have $h^0(X, L) = h^0(C_2, L_2(-\sum_{i=1}^{\delta} q_i)) \leq 2$, because $\deg L_2(-\sum_{i=1}^{\delta} q_i) = 6 - \delta$. This contradiction shows that $d_i \geq 0$ for $i = 1, 2$.

For $i = 1, 2$ we set $l_i = h^0(C_i, L_i)$ and $e_i := d_i - 2g_i$. Let

$$\epsilon := \max\{e_1, e_2, 0\} + 1 \text{ and } \beta := \min\{\epsilon, \delta\}.$$

From Addendum 3.2.2 we have

$$h^0(X, L) \leq l_1 + l_2 - \beta \leq 3. \tag{53}$$

Step 1. We exclude all the cases for which $l_1 + l_2 - \beta \leq 2$. This only requires a trivial checking. To begin with, the following cases are all excluded:

$$\delta = 6, \ \underline{g} = (0, 1), \underline{d} \in \{(0, 5), \ (3, 2), \ (4, 1), \ (5, 0)\}. \tag{54}$$

Let us just show how to treat $\underline{d} = (0, 5)$. We have $l_1 = 1$, $l_2 = 5$, $\epsilon = e_2 + 1 = 4$ and $\beta = \min\{4, 6\} = 4$. Hence $h^0(X, L) \leq 2$. All other cases are treated in the same way. If $\delta = 6$, we are left with $\underline{d} = (1, 4)$ and $\underline{d} = (2, 3)$ (of course $\underline{g} = (0, 1)$).

Let $\delta = 5$, by the same argument, we exclude

$$\delta = 5, \ \ \underline{g} = (0, 2), \ \ \underline{d} \in \{(2, 3) \ (3, 2), \ (4, 1), \ (5, 0)\} \tag{55}$$

and we exclude

$$\delta = 5, \ \ \underline{g} = (1, 1), \ \ \underline{d} \in \{(0, 5), \ (1, 4)\}. \tag{56}$$

Let $\delta = 4$. We exclude

$$\delta = 4, \ \ \underline{g} = (0, 3), \ \ \underline{d} \in \{(2, 3), \ (3, 2)\}. \tag{57}$$

and

$$\delta = 4, \ \ \underline{g} = (1, 2), \ \ \underline{d} = (4, 1). \tag{58}$$

Finally, this method applies to exclude

$$\delta = 3, \ \ \underline{g} = (0, 4), \ \ \underline{d} = (2, 3). \tag{59}$$

This finishes the list of cases for which $l_1 + l_2 - \beta \leq 2$.

From now on we always have $l_1 + l_2 - \beta = 3$ (by (53)).

Step 2. To exclude another group of cases we now use Lemma 1.1.4 and its consequence, Lemma 5.3.4. Let us begin with case $\delta = 6$, hence $\underline{g} = (0, 1)$, and $\underline{d} = (1, 4)$. In this case $\beta = 3$, so that we obviously have

$$3 = \beta < d_2 = 4 < \delta = 6. \tag{60}$$

Let $X' = (C_1 \coprod C_2)/_{\{p_i = q_i, \ \ i=1,\dots,3\}}$, let $\nu : X' \to X$ be the same map as in Lemma 5.3.4 and let $M = \nu^* L$. Then $h^0(X', M) = 3$ (by Lemma 1.1.9(ii), or by Clifford). By (60) Lemma 5.3.4 applies, yielding that $h^0(X, L) < 3$, a contradiction.

• By (54) if $\delta = 6$ the only remaining case is $\underline{d} = (2, 3)$. (V) is proved.

The previous argument can be repeated every time we have $\beta < d_i < \delta$ for some i, enabling us to exclude the following cases.

$\delta = 5$, $\underline{g} = (0, 2)$ and $\underline{d} = (1, 4)$. (Here $2 = \beta < d_2 = 4 < \delta = 5$.)

$\delta = 5$, $\underline{g} = (1, 1)$ and $\underline{d} = (2, 3)$. (Here $2 = \beta < d_2 = 3 < \delta = 5$.)

$\delta = 4$, $\underline{g} = (1, 2)$ and $\underline{d} \in \{(2, 3,)(3, 2)\}$ (If $\underline{d} = (2, 3)$ then $1 = \beta < d_2 = 3 < \delta = 4$; if $\underline{d} = (3, 2)$ then $2 = \beta < d_1 = 3 < \delta = 4$.)

We shall now exclude the two equal multidegree cases

$$\delta=5,\ \underline{g}=(0,2),\ \underline{d}=(0,5) \quad \text{and} \quad \delta=4,\ \underline{g}=(1,2),\ \underline{d}=(0,5),$$

with $l_1+l_2=5$. Let $X'=(C_1\coprod C_2)/(p_i=q_i, i=1,2)$ so that X' has two nodes. Let $L'\in \operatorname{Pic} X'$ be the pull back of L. Then $h^0(X',L')=3$, so, for $h^0(X,L)=3$ we must have $q_i\sim_{L'} p_i$ for $i\geq 3$. Now, by Lemma 1.1.4, this implies that $L_2(-q_1-q_2)$ has at least two base points, which is clearly impossible.

• By Step 2, (55) and (56) there are no more cases with $\delta=5$.

Step 3. Now we shall use Corollary 1.2.2 to exclude all the cases for which $l_1+l_2=4$ and there is $i\in\{1,2\}$ such that $l_i\geq 2$ and $\delta>\operatorname{Cliff} L_i+2$. This amounts to the following list of cases.

$\delta=4$, $\underline{g}=(0,3)$ and $\underline{d}=(0,5)$. $l_2=3$ and $\operatorname{Cliff} L_2=1$.

$\delta=4$, $\underline{g}=(1,2)$ and $\underline{d}=(1,4)$. $l_2=3$ and $\operatorname{Cliff} L_2=0$.

By the previous step and (57) the only case left with $\delta=4$ is $\underline{g}=(0,3)$ and $\underline{d}=(1,4)$. Now $\beta=2$, therefore (as $l_1+l_2-2=3$ by (53)) we have $l_2=3$, i.e. L_2 is the canonical bundle of C_2. To prove that $L_2=\mathcal{O}_{C_2}(\sum_1^4 q_i)$ it suffices to prove that $L_2(-q_1-q_2)$ has q_3 and q_4 as base points (and note that we are free to permute the q_i). We argue as at the end of Step 2: let $X'=(C_1\coprod C_2)/_{(p_i=q_i,i=1,2)}$ and let L' be the pull back of L to X'. Then $h^0(X',L')=3=h^0(X,L)$, so, $L_2(-q_1-q_2)$ has q_3 and q_4 as base points.

• (IV) is proved.

$\delta=3$, $\underline{g}=(1,3)$. We exclude $\underline{d}=(1,4)$ (as $l_2=3$ and $\operatorname{Cliff} L_2=0$), and $\underline{d}=(2,3)$ (as $l_1=2$ and $\operatorname{Cliff} L_1=0$).

$\delta=3$, $\underline{g}=(2,2)$. We exclude $\underline{d}=(1,4)$ (as $l_2=3$ and $\operatorname{Cliff} L_2=0$), and $\underline{d}=(2,3)$ (as $l_1=2$ and $\operatorname{Cliff} L_1=0$).

Step 4. From now on we assume $\delta=3$.

Let $\underline{g}=(2,2)$ and $\underline{d}=(2,3)$. Now $l_1+l_2=4$ if and only if $L_2=H_{C_2}(p)$. So L_2 has a base point, which is impossible by hypothesis. By Step 3, there are no more balanced multidegrees to treat, when $\underline{g}=(2,2)$.

Let $\underline{g}=(0,4)$. By (57) there are two cases to rule out: $\underline{d}=(0,5)$ and $\underline{d}=(1,4)$

Let $\underline{d}=(0,5)$. As $l=3$ we have $l_1+l_2=1+3=4$. It is clear that Lemma 1.1.6 applies, giving $q_1\sim_{L_2} q_2\sim_{L_2} q_3$. Therefore, if $1\leq i\neq j\leq 3$

$$2=h^0(C_2,L_2(-q_i))=h^0(C_2,L_2(-q_i-q_j))=h^0(C_2,L_2(-q_1-q_2-q_3)).$$

But then C_2 is hyperelliptic ($\deg L_2(-q_1-q_2-q_3)=2$), which implies that L_2 has a base point. A contradiction.

Let $\underline{d}=(1,4)$. As $\beta=2$ and $l=3$ we have $l_1+l_2=2+3$, so C_2 is hyperelliptic and $L_2=H_{C_2}^{\otimes 2}$. Consider $X'=(C_1\coprod C_2)/_{(p_i=q_i,\ i=1,2)}\xrightarrow{\nu} X$ and let $M=\nu^*L$. Then $h^0(X',M)=3$, therefore $p_3\sim_M q_3$. By Lemma 1.1.4 we obtain that q_3 is a base point of $L_2(-q_1-q_2)$, hence (permuting the gluing points) $H_{C_2}\neq\mathcal{O}_{C_2}(q_i+q_j)$ for all $i\neq j$. So, $L_2(-q_1-q_2)=\mathcal{O}_{C_2}(q_1'+q_2')$ where q_1' is conjugate to q_1 under the hyperelliptic series, and the same for q_2',q_2. But then, as q_3 is a base point of $L_2(-q_1-q_2)=\mathcal{O}_{C_2}(q_1'+q_2')$, we get that (say) $q_3=q_1'$, which is a contradiction.

• By Step 3, the remaining cases with $\delta=3$ have $\underline{g}=(1,3)$ and either $\underline{d}=(3,2)$ or $\underline{d}=(0,5)$. This is (III). ■

Lemma 5.3.4. *Let δ and β be two positive integers with $\delta>\beta$. Consider the partial normalization of X defined as follows*

$$X'=(C_1\coprod C_2)/_{\{p_i=q_i,\quad i=1,\dots,\beta\}}\xrightarrow{\nu} X=(C_1\coprod C_2)/_{\{p_i=q_i,\quad i=1,\dots,\delta\}}.$$

For $i=1,2$, pick $L_i\in\operatorname{Pic} C_i$ and $M\in\operatorname{Pic}(X')$ such that $M_{|C_i}=L_i$.

If $\beta<\deg L_i<\delta$ for some i, then $h^0(X,L)<h^0(X',M)$ for every $L\in F_M(X)$.

Proof. We argue by contradiction, as follows. We prove that if $\beta < \deg L_1$, and if there exists $L \in F_M(X)$ such that $h^0(X, L) = h^0(X', M)$, then $\deg L_1 \geq \delta$.

Let such an L be fixed. By Lemma 1.1.5 we have $p_i \sim_M q_i$ for all $i = \beta+1, \ldots, \delta$. Now Lemma 1.1.4 yields that, for all $i \geq \beta+1$, p_i is a base point of $L_1(-\sum_{j=1}^{\beta} p_j)$.

As $\deg L_1 > \beta$, $\deg L_1(-\sum_{j=1}^{\beta} p_j) \geq 1$. Now, a line bundle of positive degree can have at most as many base points as its degree. We just proved that $L_1(-\sum_{j=1}^{\beta} p_j)$ has $\delta - \beta$ base points, hence $\deg L_1 - \beta \geq \delta - \beta$, i.e. $\deg L_1 \geq \delta$. We are done. ■

Proposition 5.3.5. *With the set up of 5.3.1, let $X = C_1 \cup C_2$ be semistable of genus 6, with $C_1 \cdot C_2 = 1$, and let $\underline{d} \in B_5(X)$.*

There exists a globally generated $L \in W^2_{\underline{d}}(X)$ if and only if C_2 is not hyperelliptic and one of the following cases occurs.

(1) $\underline{g} = (1, 5)$, $\underline{d} = (0, 5)$, and $L = (\mathcal{O}_{C_1}, L_2)$ for some $L_2 \in W^2_5(C_2)$.
(2) $\underline{g} = (2, 4)$ or $\underline{g} = (3, 3)$, $\underline{d} = (2, 3)$, C_1 is hyperelliptic and $L = (H_{C_1}, L_2)$ for some $L_2 \in W^1_3(C_2)$.

Proof. As X is semistable we have $g_1 \geq 1$. If L is globally generated, so are L_1 and L_2; hence if $h^0(X, L) = 3$ we have $3 = l_1 + l_2 - 1$ by Lemma 1.1.5. Therefore $l_1 + l_2 = 4$.

Case $\underline{g} = (1, 5)$. The balanced multidegrees are $(0, 5)$ and $(1, 4)$. If $\underline{d} = (1, 4)$ and $l_1 = 1$ then L_1 has a base point, which is not possible. If $l_1 = 0$ then $h^0(X, L) \leq 2$. So $\underline{d} = (1, 4)$ is ruled out.

Assume $\underline{d} = (0, 5)$. By the initial observation, we must have $L_1 = \mathcal{O}_{C_1}$, $l_2 = 3$ and L_2 free from base points, hence C_2 is not hyperelliptic. Conversely, if $L_2 \in W^2_5(C_2)$ then L_2 is globally generated, because C_2 is not hyperelliptic; let $L = (\mathcal{O}_{C_1}, L_2)$ then obviously $h^0(X, L) = 3$.

Case $\underline{g} = (2, 4)$. The balanced multidegrees are $(1, 4)$ and $(2, 3)$. We rule out $\underline{d} = (1, 4)$ just as in the previous case. Assume $\underline{d} = (2, 3)$; as $l_i \leq 2$ we have $l_1 = l_2 = 2$ and C_2 cannot be hyperelliptic (for otherwise L_2 has a base point). The converse is easily proved as before.

Case $\underline{g} = (3, 3)$. This case is symmetric, so it suffices to consider the balanced multidegree $\underline{d} = (2, 3)$. We will show that C_1 is hyperelliptic and that C_2 is not. If C_1 is not hyperelliptic, then $l_1 \leq 1$; as $l_2 \leq 2$ to have $h^0(X, L) = 3$ both L_1 and L_2 must have a base point at the attaching point, which is not possible. So C_1 must be hyperelliptic. The rest of the argument is exactly as in the previous case. ■

Proposition 5.3.6. *With the notations of 5.3.1, let $X = C_1 \cup C_2$ be of genus 6 with $C_1 \cdot C_2 = 2$, and let $\underline{d} \in B_5(X)$. There exists a globally generated $L \in W^2_{\underline{d}}(X)$ if and only if one of the following cases occurs.*

(1) $\underline{g} = (0, 5)$, $\underline{d} = (1, 4)$, C_2 hyperelliptic and $L_2 = H_{C_2}^{\otimes 2}$.
(2) $\underline{g} = (1, 4)$, $\underline{d} = (0, 5)$, C_2 non-hyperelliptic, $L_1 = \mathcal{O}_{C_1}$, $h^0(L_2) = 3$ and $h^0(L_2(-G_2)) = 2$.
(3) $\underline{g} = (1, 4)$, $\underline{d} = (2, 3)$, $L_1 = \mathcal{O}_{C_1}(G_1)$, C_2 non-hyperelliptic, $L_2 = \mathcal{O}_{C_2}(G_2+q)$, $h^0(L_2) = 2$.
(4) $\underline{g} = (2, 3)$, $\underline{d} = (2, 3)$, $H_{C_1} = \mathcal{O}_{C_1}(G_1) = L_1$, C_2 non-hyperelliptic and $L_2 = \mathcal{O}_{C_2}(G_2 + q)$, $h^0(L_2) = 2$.

Proof. Notice that, as L has no base points, L_1 and L_2 have no base points.

Let $\underline{g} = (0, 5)$ and $\underline{d} = (1, 4)$ (this is a strictly semistable curve and C_1 its exceptional component). By Lemma 1.1.9 we have $l \leq l_1 + l_2 - 2 \leq 2 + 3 - 2 = 3$, and equality holds if and only if $l_2 = 3$, if and only if C_2 is hyperelliptic and $L_2 = H_{C_2}^{\otimes 2}$, as stated. It is clear that every L pulling back to $(\mathcal{O}(1), H_{C_2}^{\otimes 2})$ on the normalization of X has $h^0(L) = 3$.

If $g_1 \geq 1$, one checks easily (by Proposition 1.2.3 and the fact that L_1 and L_2 have no base points) that $l_1 + l_2 = 4$. Hence by Lemma 1.1.6 we have

$$p_1 \sim_{L_1} p_2 \text{ and } q_1 \sim_{L_2} q_2, \tag{61}$$

and L is uniquely determined by its pull-back to the normalization, by Lemma 1.1.5.

• Assume $\underline{g} = (1, 4)$. If $\underline{d} = (0, 5)$, by Proposition 1.2.3 (ii) we obtain $L_1 = \mathcal{O}_{C_1}$ and Cliff $L_2 = 1$ so $h^0(L_2) = 3$. C_2 cannot be hyperelliptic, for otherwise L_2 will have a base point. Moreover, as $q_1 \sim_{L_2} q_2$, we have

$$h^0(L_2(-q_1 - q_2)) = h^0(L_2(-q_1)) = h^0(L_2(-q_2)) = 2.$$

as claimed. The converse follows easily from Lemma 1.1.6. Suppose now $\underline{d} = (1, 4)$. As $p_1 \sim_{L_1} p_2$, we have $L_1 = \mathcal{O}_{C_1}(p)$ with $p \neq p_i$. So, L_1 has a base point in p, which is not possible. This case does not occur. Finally, let $\underline{d} = (2, 3)$. We must have $l_1 = l_2 = 2$ (as C_2 cannot be hyperelliptic, as before). By (61) we obtain $L_1 = \mathcal{O}_{C_1}(p_1 + p_2)$ and $L_2 = \mathcal{O}_{C_1}(q_1 + q_2 + q)$ for a (uniquely determined) $q \in C_2$. The converse follows from Lemma 1.1.6.

• Now assume $\underline{g} = (2, 3)$. If $\underline{d} = (2, 3)$ we argue exactly as in the previous case ($\underline{g} = (1, 4)$, $\underline{d} = (2, 3)$). If $\underline{d} = (1, 4)$ we have $l_1 = 1$ so that $L_1 = \mathcal{O}_{C_1}(p)$ with $p \neq p_i$ for $i = 1, 2$ (as $p_1 \sim_{L_1} p_2$). So L has a base point in p; this case is excluded. Finally, if $\underline{d} = (3, 2)$, arguing as before one obtains that L_1 has a base point in $p \in C_1$, impossible. This finishes all the possible cases, so we are done. ■

References

[A04] Alexeev, V.: *Compactified Jacobians and Torelli map.* Publ. RIMS, Kyoto Univ. 40 (2004), 1241-1265.

[ACGH] Arbarello, E.; Cornalba, M.; Griffiths, P. A.; Harris, J.: *Geometry of algebraic curves. Vol. I.* Grundlehren der Mathematischen Wissenschaften 267. Springer-Verlag, New York, 1985.

[B99] Bruno A.: *Degenerations of linear series and binary curves.* Ph.D. thesis (1999) Brandeis University, Waltham.

[C94] Caporaso, L.: *A compactification of the universal Picard variety over the moduli space of stable curves.* J. of the Amer. Math. Soc. **7** (1994), 589-660.

[C07] Caporaso, L.: *Geometry of the theta divisor of a compactified Jacobian.* To appear in J. of the Europ. Math. Soc. Available at arXiv: 0707.4602.

[C08] Caporaso, L.: *Brill-Noether theory of binary curves.* To appear in Math. Res. Let. Available at arXiv: 0807.1484.

[C882] Clifford, W. K.: *On the classification of loci.* In Mathematical papers. Chelsea Publishing Co., New York. 1882.

[CH88] Cornalba, M.; Harris, J.: *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves.* Ann. Sci. École Norm. Sup. (4) 21 (1988), no. 3, 455-475.

[EH86] Eisenbud, D.; Harris, J.: *Limit linear series: basic theory.* Invent. Math. 85 (1986), no. 2, 337-371.

[EM02] Esteves E.; Medeiros N.: *Limit canonical systems on curves with two components.* Invent. Math. 149 (2002), no. 2, 267-338.

[HM] Harris, J.; Morrison, I.:*Moduli of curves.* Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998.

[HM82] Harris, J.; Mumford, D.:*On the Kodaira dimension of the moduli space of curves.* Invent. Math 67 (1982), 23-86.

[OS79] Oda, T.; Seshadri, C.S.: *Compactifications of the generalized Jacobian variety.*Trans. A.M.S. 253 (1979) 1-90.

[O06] Osserman, B.: *A limit linear series moduli scheme.* Ann. Inst. Fourier (Grenoble) 56 (2006), no. 4, 1165-1205.

[P96] Pandharipande, R.: *A compactification over $\overline{M_g}$ of the universal moduli space of slope-semistable vector bundles.* Journ. of the Amer. Math. Soc. 9(2) (1996) 425-471.

[S94] Simpson, C. T.: *Moduli of representations of the fundamental group of a smooth projective variety.* Inst. Hautes Études Sci. Publ. Math., 80; 5-79, 1994.